Rhythmicon Relationships, Farey Sequences, and James Tenney's Spectral CANON for CONLON Nancarrow *(1974)**

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Multi-layered polyrhythms have appeared frequently in Western music of the past century. Such rhythmic combinations can display surprising, emergent forms of perceptible order. In this essay, I explore the objective and perceived characteristics of two such stratified rhythmic structures. The first of these, a *divisive polyrhythmic array*, superimposes "tuplets" of progressively higher order, each *subdividing* a common basic duration. In the second structure, a *multiplicative polyrhythmic array*, the inter-attack durations in successive rhythmic layers are progressively higher *multiples* of a common basic duration. Each of these array types exhibits characteristic composite features that are readily audible in performances and visible in scores, including patterns of attack coincidences and arpeggiations traversing multiple rhythmic layers. Mathematical modeling of the arrays elucidates such features. I conclude with a detailed analysis of a composition by James Tenney in which polyrhythmic arrays of both sorts supply fundamental musical material, and whose formal design involves a transition between the two array types. A detailed correspondence between pitch and rhythmic structures in that work is also examined.

Keywords: canon, Cowell, Farey, polyrhythm, rhythmicon, Tenney

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INTRODUCTION

Example 1 reproduces two diagrams from Henry Cowell's seminal text *New Musical Resources*.[1] The one on the left illustrates how the wavelength of the *n*th superimposed partial of a harmonic complex tone evenly divides the wavelength of its fundamental into *n* parts.[2] With the diagram on the right, Cowell proposed analogously superimposing subdivisions of a rhythmic value. He later co-developed with inventor Léon Theremin a mechanical device—dubbed the *rhythmicon*—capable of accurately executing music whose pitches and tempi each displayed such multi-layered rational relationships.[3] As a consequence, these sorts of layered divisive polyrhythmic structures are sometimes referred to as *rhythmicon relationships*.

Often reflecting Cowell's influence, stratified polyrhythmic structures have since assumed a vast array of compositional manifestations, especially in North America.[4] Perhaps the most widely known and varied examples are found in Conlon Nancarrow's fifty-one *Studies for Player Piano*, most of which employ complex rational tempo relationships.[5] Further examples appear in the music of composers as stylistically diverse as Elliott Carter,[6] Ben Johnston,[7] Larry Polansky, John Luther Adams, David First, and many of the New York-based "totalist" composers of the 1980s.[8] (Example 2[a]–[c] shows some instances.) Diagrammatic representations of such structures appear in the writings of Karlheinz Stockhausen[9] and Carter.[10] More recently, composers Philip Blackburn, Janek Schaefer, Matthew Burtner,[11] Annie Gosfield, Nick Didkovsky, Robert Normandeau, and others have imagined new ways to use *rhythmicon relationships*.[12] Jim Bumgardner's online "Whitney Music Box" is an elegant example of an audio-video rhythmicon.[13] Readers may wish to try creating custom rhythmicon structures themselves using the user-programmable "Online Rhythmicon," created by Didkovsky.[14]

* I thank Dmitri Tymoczko for his helpful responses to earlier versions of this paper and Clarence Barlow for making available his recording of the extended version of Tenney's *Spectral CANON*.

1 Cowell ([1930] 1996, 47–48).

2 A *harmonic complex tone* is a collection of sinusoids, all of whose frequencies are positive integer multiples of some *fundamental frequency* or, equivalently, all of whose wavelengths are integer subdivisions of some fundamental wavelength. Most definitely pitched musical tones are well modeled as harmonic complex tones. See Roederer (1995, 60).

3 Smith (1973, 134).

4 Other catalysts for the appearance of complex polyrhythms in contemporary Western music have included the influence of non-Western musics, late medieval Western music, observations of coincident but independent sound sources, and various structuralist approaches to rhythm. The structures examined in this paper are ones that relate particularly closely to Cowell's analogy from the makeup of complex tones.

5 Gann (1995, 5–8).

6 Bernard (1988); Carter (1977, 292–94).

7 Von Gunden (1986, 68–70).

8 Gann (1997, 189) and (2006, 127–29).

9 Stockhausen (1957 [1959], 17, 27).

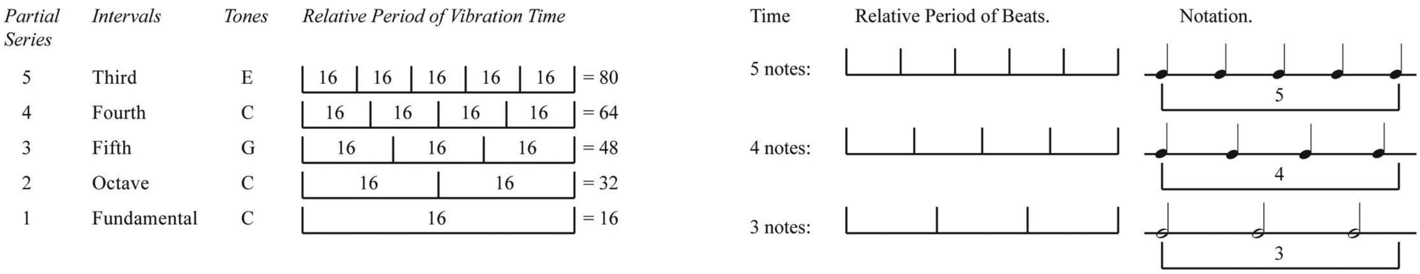
10 Carter (1977, 294).

11 Burtner (2005).

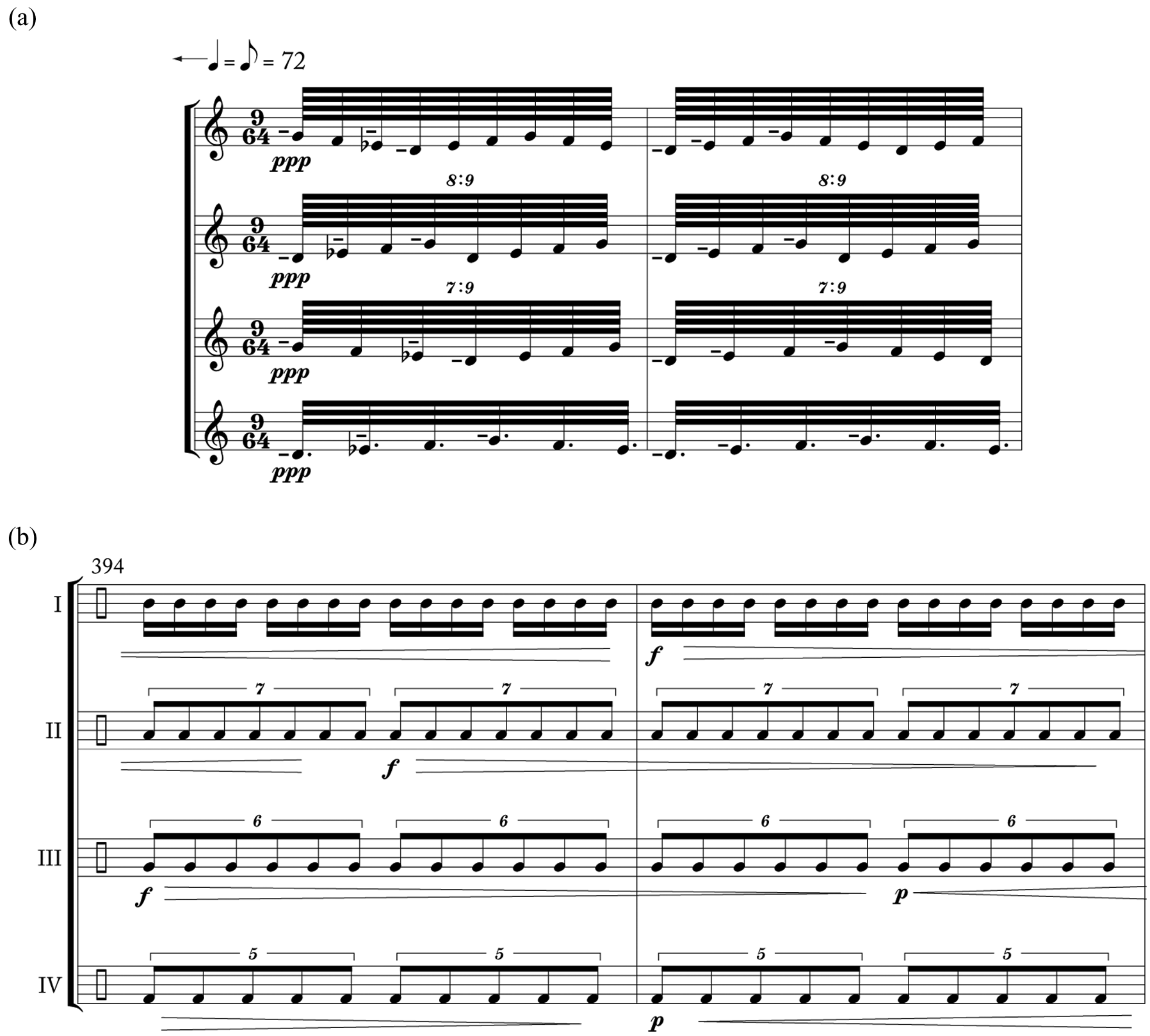
12 *The Art of the Virtual Rhythmicon* (2006).

13 Bumgardner (n. d.). For further documentation, see Bumgardner (2009).

14 The Online Rhythmicon (n. d.).



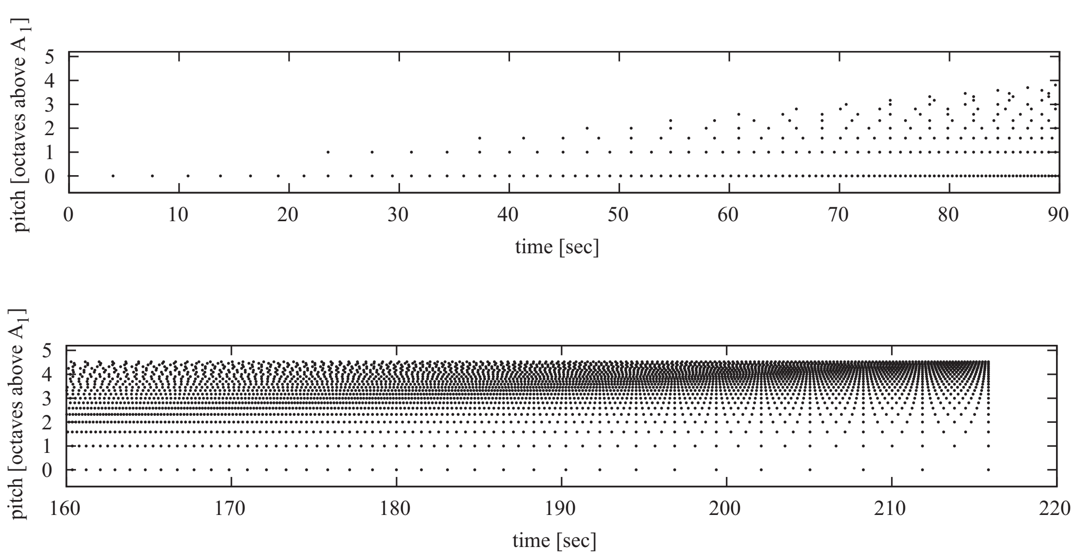
EXAMPLE 1. *Cowell's analogy between the wavelengths of harmonic partials and the regular rhythmic subdivisions of a musical duration, as illustrated by two figures (redrawn) from Cowell ([1930] 1996). The typesetting and alignment of elements follow those found in Cowell's book. Reprinted with the permission of Cambridge University Press.*



EXAMPLE 2. *Multilayered divisive polyrhythms appearing in selected compositions: (a) Ben Johnston, String Quartet No. 4 (1973). Copyright 1990 by Smith Publications, Sharon, VT. All rights reserved. Used by permission. (b) John Luther Adams,* Earth and the Great Weather *(1993). Copyright 1993 by Taiga Press, Fairbanks, AK. All rights reserved. Used by permission. (c) Elliott Carter, String Quartet No. 5 (1995). Copyright 1995 by Boosey and Hawkes, New York, NY. All rights reserved. Used by permission.*

(c)

example 2. [Continued]



example 3. *Graphic scores for the (a) beginning and (b) end of James Tenney's* Spectral CANON for CONLON Nancarrow. *(See* MTS *online to listen to the original 1974 recording of the complete work.)*

Particularly clear examples of such polyrhythmic structures appear in the music of American-Canadian composer James Tenney (1934–2006), in works such as his *Spectral CANON for CONLON Nancarrow* (1974) for harmonic player-piano; *Three Harmonic Studies*, III (1974), for small orchestra; *Septet* (1981/1996) for six electric guitars and electric bass; *Spectral Variations I–III* (1991/1998) for computer-driven piano; and *Song 'n' Dance for Harry Partch*, II: "Mallets in the Air" (1999), for adapted viola, diamond marimba, strings, and percussion.[15]

Example 3 presents graphic scores for the opening and conclusion of Tenney's *Spectral CANON for CONLON Nancarrow*

15 Readers interested in a more general introduction to Tenney and his work may wish to consult Polansky (1983) and Hasegawa, ed. (2008).

with dots representing attacks. Despite the gradual *ritardando* evident in Part (b), it is clear that at the work's close the reiterations of any given pitch evenly subdivide those of the lowest pitch, thus displaying the classic rhythmicon relationships illustrated in Example 1. I will henceforth refer to this particular type of multi-voice rhythmic structure as a *divisive (polyrhythmic) array*.

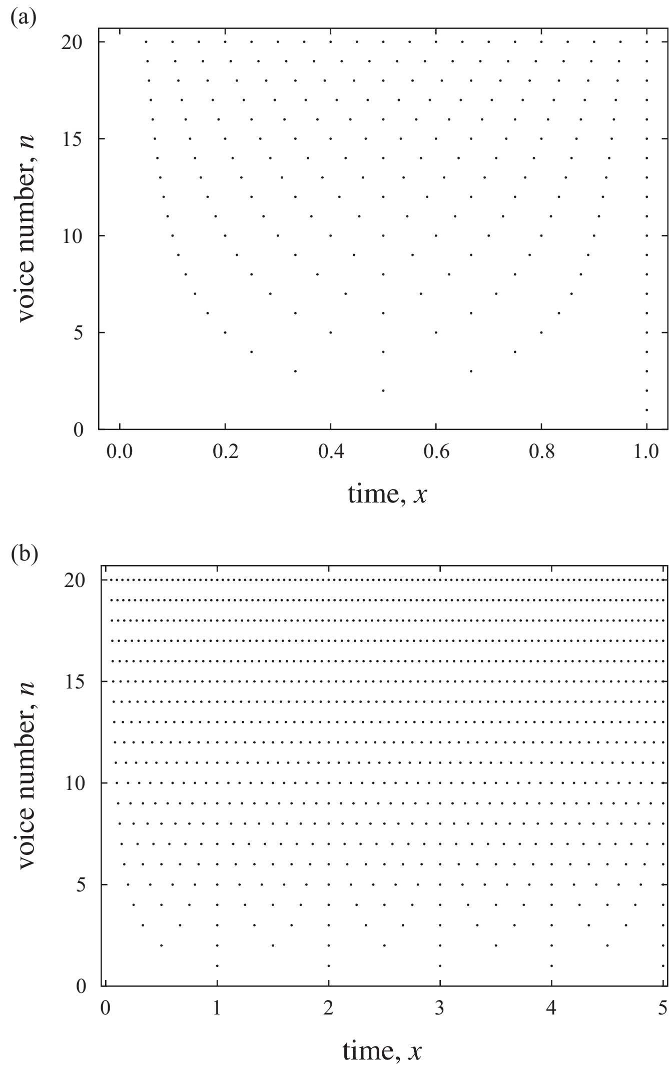
Part (a) of Example 3 exhibits a structure different from any of those mentioned above, wherein the time-interval between two successive attacks of any given pitch is a fixed multiple of the inter-attack duration for the lowest pitch. I refer to such a rhythmic structure as a *multiplicative (polyrhythmic) array*. Multiplicative arrays, unlike divisive ones, appear only occasionally in the musical literature; for instance, Carter diagrams the rhythmic structure of the introduction to his *Double Concerto* (1961) as the superimposition of a partial divisive array and a partial multiplicative array.[16] Other examples include Philip Corner's *Gamelan II* (1975),[17] and Carter Scholz's *Rhythmicon I* (1988).[18]

In this essay, I will elucidate features of divisive and multiplicative polyrhythmic arrays as paradigmatic structures capable of various compositional manifestations. I begin by introducing appropriate theoretical machinery for the description of such textures in Sections 1 and 2 below. Section 3 applies the resulting conceptual framework in a detailed analysis of Tenney's *Spectral CANON for CONLON Nancarrow*. Supporting mathematical details that are inessential to the exposition have been placed in the Appendix.

1. DIVISIVE ARRAYS

Example 4 illustrates a canonical divisive array on two different ranges of the horizontal axis, showing that the array is periodic with a period of one. Elapsed time, x, increases from left to right in the examples, with each dot representing a sonic attack. The units are arbitrary. Twenty *voices* in the array are represented by horizontal strata and indexed by a *voice number*, n, ranging from 1 to 20.[19] The location in time of an attack depends both on n and on its *attack number*, m, which is its order number within that voice. (In other words, reading from left to right, $m = 1$ for the first attack within any given voice, $m = 2$ for the second attack, etc.) Since the nth voice subdivides the unit of time into n equal durations, the time of the mth attack within that voice is

$$x(m,n) = m/n,$$



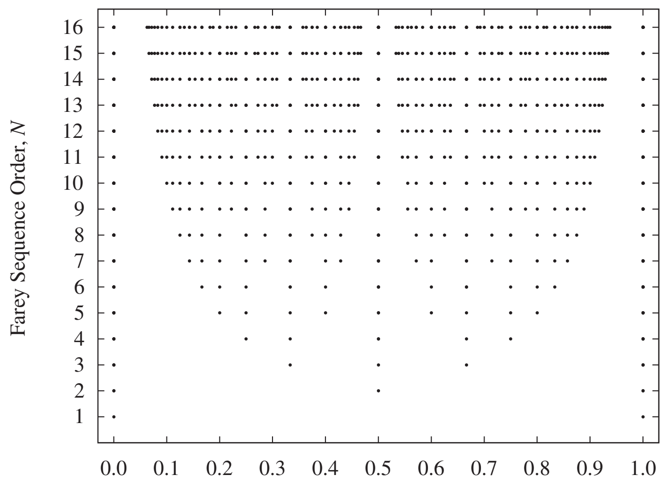
EXAMPLE 4. *Canonical divisive polyrhythmic array: (a) between 0 and 1; (b) between 0 and 5.*

which I will take to be the *defining equation* of a canonical divisive array.

Consider a time $x(m,n) = m/n$, where m/n is a reduced rational number.[20] For any positive integer k, $x(km, kn) = km/kn = m/n = x(m,n)$ so that voice number kn also attacks at time x. That is, n is the lowest voice number among the voices attacking at time x and the other voices attacking at that time are precisely those whose voice numbers are multiples of n. In particular, when Voice 1 attacks, all other voices attack simultaneously with it.

16 Carter (1977, 294).
17 Lely (2012, 172, 174).
18 Schneider (2003).
19 The structural correspondence to the conclusion of Tenney's *Spectral CANON* is visible (cf. Example 3[b]) since, in that composition, each voice reiterates only a single pitch. Strictly speaking, however, pitch is not represented in Example 4 since, in other musical contexts, any single voice may present a variety of pitches.

20 A rational number is any number expressible as a quotient of two integers (i.e., a fraction) with the denominator not equal to zero. A rational number is said to be in *lowest terms* or *reduced form* when its numerator and denominator have no common divisors greater than one (i.e., where all common factors have been "canceled" from the numerator and denominator).



EXAMPLE 5. *Farey sequences of order one through sixteen plotted as points.*

1.1. RHYTHMIC FAREY SEQUENCES

The *Farey sequence* of order N is the increasing sequence F_N of all reduced rational numbers in the interval from 0 to 1 having denominators less than or equal to N. The first few Farey sequences are listed below, and the sixteen lowest-order Farey sequences are illustrated in Example 5.

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$

$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$

An *extended Farey sequence* is the periodic extension of a Farey sequence into the non-negative rational numbers. For instance, the extended Farey sequence corresponding to F_2 is

$$\left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \ldots\right\}.$$

The Nth order extended Farey sequence can be thought of as the aggregate sequence of all attack times in the lowest N voices of a canonical divisive array (or, referring to Example 4, as the projection onto a horizontal axis of all attacks in those voices).[21]

21 Strictly speaking, an extended Farey sequence contains a term at zero, whereas the divisive polyrhythmic arrays of Example 4 do not (for reasons of convenience) in the musical analyses to follow.

A comparison between Examples 4 and 5 should help to make this clear.

The terms in a Farey sequence exhibit a number of interesting regularities and interrelationships, but I will mention (without proof) only the few that are needed here.[22] First, it is the case that no two consecutive Farey fractions have the same denominator. Next, consider three consecutive reduced Farey fractions in F_N: $m_1/n_1 < m_2/n_2 < m_3/n_3$. Then

$$\frac{m_2}{n_2} = \frac{m_1 + m_3}{n_1 + n_3},$$

which is the *Farey mediant* of m_1/n_1 and m_3/n_3. Moreover, F_N can be recursively constructed by inserting between each pair of successive terms in F_{N-1} their Farey mediant whenever its denominator is no greater than N. Thus F_4 can be constructed from F_3 by inserting $1/4$ between $0/1$ and $1/3$, and also inserting $3/4$ between $2/3$ and $1/1$, but omitting the Farey mediant $2/5$ between $1/3$ and $1/2$ and the Farey mediant $3/5$ between $1/2$ and $2/3$ (although these omitted fractions find a rightful home in F_5, as shown in Example 5).

Now consider a Farey fraction m_0/n_0 in F_N, its two immediate neighbors in F_N, and the Farey mediants between it and each of those neighbors. The interval between the two mediants is called a *Farey arc*. It obviously includes the Farey fraction in question but no others in F_N, while the collection of these non-overlapping Farey arcs covers the interval from 0 to 1. It is further possible to establish bounds on the length of each Farey arc.[23] For $N > 1$, each side of the arc containing the Farey point m_0/n_0 has a length between

$$\frac{1}{n_0(2N-1)} \quad \text{and} \quad \frac{1}{n_0(N+1)}.$$

The lower bound is particularly significant since it shows that, for a given N, reduced fractions of a lesser denominator n_0 are better separated from their neighbors than reduced fractions of a relatively greater denominator. Martin Huxley offers the vivid analogy of gravitational lensing "in which the images of faint stars are displaced away from a nearby bright star."[24] This property of the rational number system is visible in both Examples 4 and 5 where, for instance, Farey fractions with denominators less than or equal to 3 (e.g., 1/1, 1/3, 1/2, 2/3, etc.) appear flanked by particularly sizable empty "buffer zones."[25]

22 Readers interested in a rigorous dedicated treatment of Farey sequences may wish to consult Hardy and Wright ([1938] 1979, 23–31).

23 Ibid. (30).

24 Huxley (1996, 8).

25 In addition to its relevance for this rhythmic analysis, the boundedness property of Farey arcs has important implications for harmonic perception, since it implies that reduced frequency ratios of relatively low denominator (i.e., strong sensory consonances) tend to be comparatively well separated from one another in the set of all possible frequency ratios ordered by size. For example, there is no other ratio of comparable consonance in the immediate neighborhood of either a 2:1 octave or a 3:2 just fifth.

EXAMPLE 6. *Measures 17–20 of* Septet *(1981/1996) by James Tenney. Copyright 1996 by Sonic Art Editions. All rights reserved. Used by permission of Smith Publications, Sharon, VT. (See* MTS *online to listen to this excerpt.)*

In rhythmic terms, an extended Farey sequence of order N is generated by a polyrhythm comprising superimposed tuplets of all orders up to and including N-tuplets. If more than a few voices are present, and the unit time-duration is in the range of 2–8 seconds, the relatively large number of voices attacking concurrently at times corresponding to reduced fractions with low denominators usually makes audible some relatively simple polyrhythmic accentuation patterns within even a very complex aggregate texture. Attacks falling on less strongly accented time-points tend to create a superimposed stochastic rhythm, but fluctuations in the attack density impart noticeable rhythmic qualities.

For large N, the edges of the larger Farey-arc "buffer zones" are delineated by closely spaced attack successions whose temporal density decreases away from these zones, typically producing an accelerating attack sequence before the buffer zone and a decelerating sequence after it. The ends of these *accelerandi*, and the beginnings of the *ritardandi*, may supply points of rhythmic emphasis. Particularly salient are the attacks delineating the large buffer zones near integer time-values. For instance, when the array is limited to $N = 7$ voices it is possible to detect a septuplet pulse emphasized by the attacks at $\{1/7, 6/7, 7/7, 8/7, 13/7, 14/7, 15/7, \ldots\}$.

Rhythmic Farey sequences figure in James Tenney's *Septet* for six electric guitars and electric bass.[26] The work is divided into five sections. The first of these is a seven-voice rhythmic canon on a single pitch (a sounding A_2). At the outset, successive voices enter at intervals of two measures. Each voice progresses step by step from simple to more complex tuplets, changing every two measures so as to divide the constant measure-lengths into successively shorter durations. Example 6 shows the resultant texture in mm. 17–20.

The uppermost part (Guitar 1) leads the rhythmic canon with the lower ones following in turn, each effectively passing its rhythm to the instrument below it in the score at two-measure intervals. The resulting aggregate attack patterns are Farey sequences of steadily increasing order. Example 6 shows that they are eighth order in mm. 17–18 and ninth order in mm. 19–20. The composer has omitted attacks in faster-moving voices whenever they would coincide with an attack in a slower-moving voice. This prevents the simpler embedded tuplets from being accentuated by attack coincidences, thus increasing uniformity in the rhythmic texture. Additionally, the *tutti* attack at

26 *Septet* (1981/1996) is recorded on Josel (1998), and also on *Cocks Crow, Dogs Bark: New Compositional Intentions* (1997).

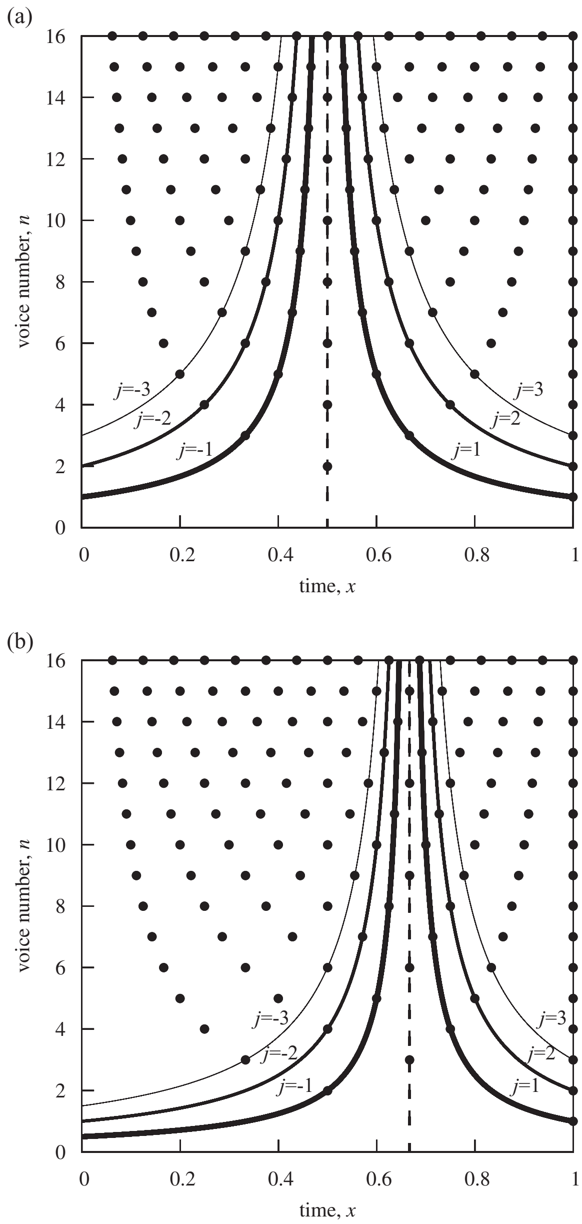
the beginning of each measure—where the downbeat would otherwise likely be heard—is omitted entirely, further heightening the ambiguity of the rhythm so that the perceived downbeat can be heard at either the beginning of a measure or at the first attack in the measure. In either event, at the marked tempo the attack *ritardandi* and *accelerandi* flanking barlines are audible with less-orderly stochastic rhythms intervening.[27]

1.2. flanking curves and outside-voice profiles

Additional considerations arise when the individual voices of a divisive array are aurally discriminable (due, for instance, to registral segregation). Noteworthy features of the canonical divisive arrays in Example 4 include the families of rising and falling curves visible near integer values of the time-coordinate. Close inspection of the examples reveals that the array points also describe families of such "flanking curves" about *any* rational-valued vertical asymptote. The family around time-value $x = 1/2$ is illustrated in Example 7(a), while Example 7(b) reveals the less conspicuous family around $x = 2/3$. The points on these flanking curves near their asymptotes correspond to the attack *accelerandi* and *ritardandi* flanking the evacuated Farey-arc buffer zones of Example 5, but here it can be seen how these rhythmic figures glissade across distinct voices and how flaring buffer zones broaden with decreasing voice number.

Explicit formulae for these flanking curves and the Farey points thereon are derived in the Appendix. Here it will suffice to note a few of their salient properties. Observe, for instance, that the flanking curves about $x = 2/3$ are steeper than those about $x = 1/2$, which in turn are steeper than those about $x = 0$. In fact, it is shown in the Appendix that the larger the asymptote's denominator n_0, the steeper will be the flanking curves about it (i.e., the more rapid will be the *glissandi*).

Examination of Example 7 reveals that not every voice necessarily attacks on a given flanking curve. In fact, it is shown in the Appendix that, for a flanking curve about asymptote m_0/n_0, only one out of every n_0 values of n will be associated with a point on the curve. Correspondingly, the most strikingly visible curves in Example 4(a) flank integer values of x (i.e., values for which $n_0 = 1$) since every voice attacks on each of these curves. On the other hand, it can be seen in Example 7 that, on any given curve flanking $x = 1/2$, only every second voice attacks (since the asymptote's reduced denominator is $n_0 = 2$), while on a curve flanking $x = 2/3$ only every third voice attacks.

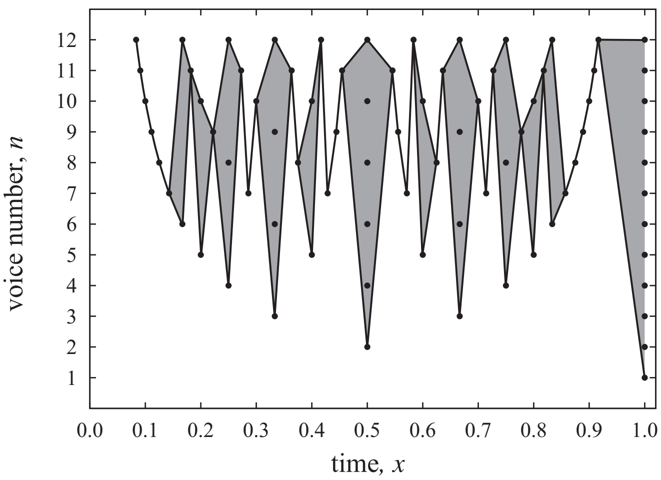


example 7. *Families of flanking curves embedding the canonical divisive array about rational abscissae at (a) 1/2, and (b) 2/3; the integer* j *indexes distinct curves.*

Furthermore—and importantly for Section 1.4 to follow—over successive flanking curves the set of attacking voices exhibits a cyclical pattern with period n_0. For instance, Example 7(a) shows that even-numbered voices attack on curves with even indices j, while odd-numbered voices attack on curves with odd indices j; i.e., the pattern of attacking voices is cyclical with period $n_0 = 2$.

An elegant characteristic of divisive arrays is the manner in which different flanking curves share attacks. In Example 7(a), for instance, the descending curves flanking $x = 0$ can be seen to share attacks with the ascending curves flanking $x = 1$, as well as

27 Note that in mm. 19–20 an aggregate attack pattern is articulated that represents a complete ninth-order Farey sequence (apart from the omitted *tutti*s), even though only seven instrumental parts are employed. This is accomplished by having Guitar 6 in mm. 19–20 play two canonic voices: the attack time-point union of the Guitar 5 and Bass parts from mm. 17–18. Thus Guitar 6 plays not only the attacks that Guitar 5 played in the previous two measures, but also those attacks that Guitar 5 "omitted" in order to avoid coincidences with the bass. This approach is extended to the end of the section (m. 26), where a complete twelfth-order Farey sequence is presented.



EXAMPLE 8. *A divisive array with twelve voices showing profiles of highest and lowest voices with the area between them filled in gray.*

with the curves (both ascending and descending) flanking $x = 1/2$. Moreover, the Appendix shows that every descending curve asymptotic to a Farey fraction in F_N intersects at an attack-point with every ascending curve asymptotic to the succeeding Farey fraction in F_N (whatever the value of N). This is musically significant if these curves function as auditory streams, since the ear can thus move easily from a descending-voice *glissando* to an ascending one at many different points of intersection, the timing perhaps being indeterminately subject to volition and/or happenstance.

Other aspects of a divisive array with potential perceptual significance include the sequences of highest and lowest sounding voices. The sequence of the latter in an N-voice array is the sequence of denominators in the associated Farey sequence F_N, which displays the same periodicity and symmetry as the Farey sequence itself. The sequence of highest sounding voices is that of the greatest multiples (not exceeding N) of the Farey sequence denominators. The profiles of highest and lowest voices for a divisive array with $N = 12$ voices are shown in Example 8. The area between these profiles is filled with gray as an aid to visually discerning them and the varying distance between them. Note that for the highest points on the $j = \pm 1$ curves flanking rational abscissas of low denominator (i.e., integers and half-integers), the lowest-voice and highest-voice curves coincide for multiple successive attacks. This promotes the hearing of these *glissandi* as prominent and unambiguous features of the musical texture.

1.3. OTHER MANIFESTATIONS OF DIVISIVE ARRAYS

Divisive arrays also make appearances in connection with some well-known acoustical and psychoacoustical phenomena. Consider the natural harmonics of a string excited at one end: when the string is lightly touched at a reduced fraction $x = m/n$ of its length, it sounds its nth harmonic and all multiples thereof. Thus the canonical divisive array of Example 4(a) illustrates the locations and harmonic numbers of all natural string harmonics if the abscissa between 0 and 1 is interpreted as the spatial extent of the string (rather than musical time).

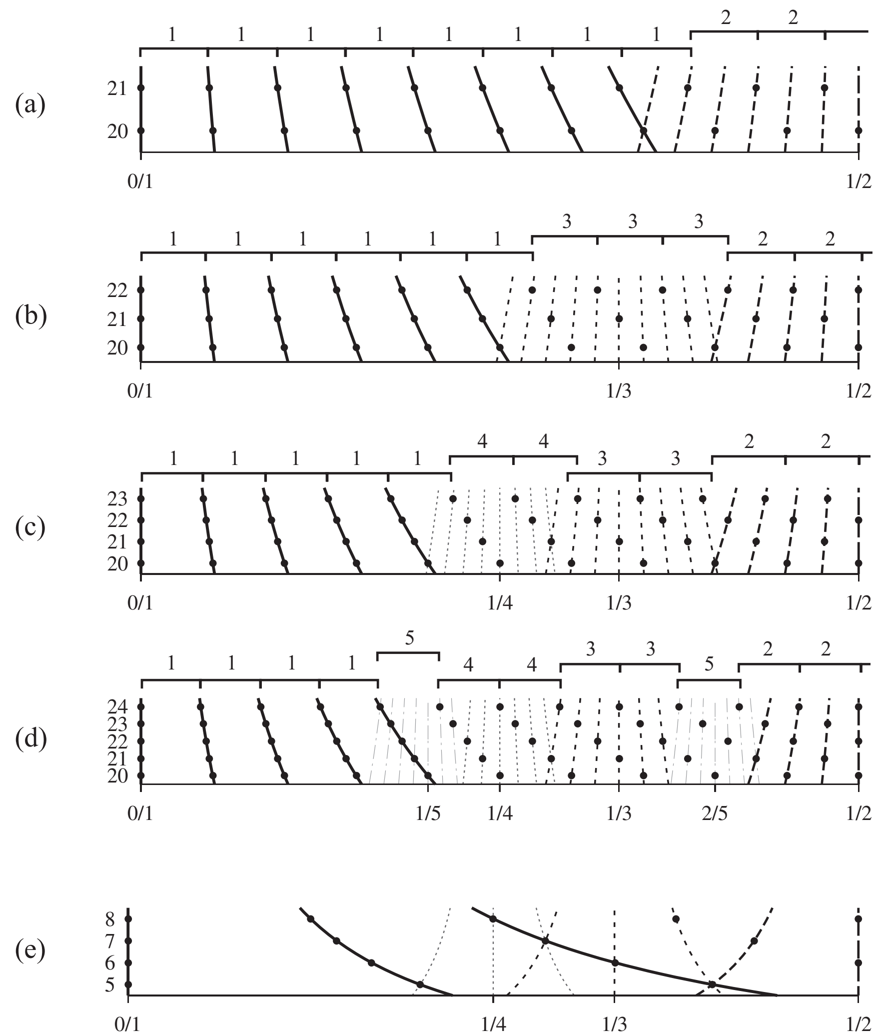
Another acoustical example involving a divisive array is provided by the combination of two complex tones having slightly different fundamental frequencies. If the frequency difference between the fundamentals is $f_2 - f_1$, then the frequency difference between nth harmonics is $nf_2 - nf_1 = n(f_2 - f_1)$. Tones acoustically beat against each other at a rate equal to the difference of their frequencies, so the nth harmonics therefore beat n times faster than the fundamentals. If we assume that all partials begin in cosine phase (i.e., with waveform crests aligned) and regard each pair of beating partials as a "voice," then the times of peak constructive interference between nth partials are described by the canonical divisive array. If the beating is sufficiently slow, the audible result is the "phaser" effect familiar to electric guitarists, with its characteristic harmonic *glissandi* across registers.

1.4. CONSECUTIVE-VOICE SUBSETS

This section presents a particular application of the formalism introduced above. The divisive arrays discussed thus far are all *complete* in the sense of containing all voices from Voice 1 through some maximum Voice N. Farey sequences and flanking curves, however, are also useful concepts for modeling certain interesting subsets of complete arrays. Example 9 shows several such structures, in which the set of voice numbers comprises consecutive integers. Only the first half of each rhythmic cycle is shown (up to time $x = 1/2$), the second half being retrograde-symmetric with respect to the first. In addition to the dots representing attacks, the figures include the visible segments of flanking curves about selected rational asymptotes.

The following discussion broaches the perceived rhythmic structure of such partial divisive arrays under the assumptions that (a) register increases with voice number, (b) the individual voices are consistently discriminable via registral separation, (c) the performance tempo is not so great that the temporal relationships between attacks in distinct voices become difficult to perceive, and (d) the total number of voices involved is small compared with their voice numbers, n. Audio mockups of these polyrhythms are easily made using commonplace notation software or Didkovsky's "Online Rhythmicon," and the reader is strongly encouraged to listen to them in conjunction with the following discussion. Listeners who have not previously heard such large-scale polyrhythms are often surprised by the internal rhythmic structures they reveal.[28]

28 For audio mockups of Example 9(a)–(d), durations of roughly 45 seconds for one complete polyrhythmic cycle are recommended. This may be compared with other durations in the range 10–80 seconds. It should be kept in mind, however, that the combination of a sufficiently fast tempo with a large registral separation between voices will make it impossible for the ear to judge the temporal relationships between attacks in distinct voices (see Moore [1997, 262]).



example 9. *Examples of consecutive-voice subsets of divisive arrays. The horizontal axis represents time* x, *while the vertical axis represents voice number* n. *The locations of important asymptotes are marked along the time-axis. (See* MTS *online to listen to Example 9[a]–[e].online.)*

It is instructive to examine how the audible rhythmic organization of the polyrhythm depicted in Example 9(c) evolves, beginning from its outset. Initially, all four voices attack in unison. As time x increases and the attack times of higher and lower voices diverge, this compound attack is repeated with a periodicity equal to that of the highest voice and with a gradually lengthening downward arpeggiation. That is, individual attacks within each arpeggio are separated by relatively small (but increasing) intragroup time-intervals. Concurrently, the end of each four-attack arpeggio is separated from the beginning of the next by a larger (but decreasing) intergroup time-interval.

As the attacks in the highest and lowest voices gradually converge in time, the intragroup and intergroup time-intervals become increasingly similar, until a threshold is reached at which the rhythm undergoes a striking qualitative transformation from a succession of composite-but-autonomous *arpeggiando* attacks (a metrical "one-feel") to a sequence of individual attacks cycling periodically across the four voices (a metrical "four-feel," with a "downbeat" imparted by the pitch accent of the uppermost voice). As the figure suggests, this change can be understood as the result of moving from a rhythmic structure governed by flanking curves about $x = 0$ to one governed by flanking curves about $x = 1/4$. The attack groups of the opening arpeggios correspond to sets of attacks residing together on single flanking curves about $x = 0$. Near the qualitative transition point, their intragroup attack-time intervals become the intergroup attack-time intervals of the new metrical structure organized about $x = 1/4$. In this new structure, each "attack group" comprises just a single attack. (In other words, there is one attack per flanking curve, since there are only four voices present and—in accordance with the discussion of Section 1.2—only every fourth voice attacks on flanking curves about any reduced rational asymptote with a denominator of $n_0 = 4$.) These time-feels, as I hear them, are indicated using brackets above the figure.

Later, attacks in the highest (fastest) and lowest (slowest) voices again approach alignment. This initiates a passage of triple meter organized about asymptote $x = 1/3$. The transition features a metrical elision, wherein the last attack of a four-pattern becomes a grace note preceding the downbeat of a new three-pattern. The location of this new downbeat coincides with the attacks of the uppermost voice. These attacks possess a pitch accent due to their height, strongly reinforced by a dynamic accent appearing in every third attack group and imparted by two near-coincident attacks. As mentioned in Section 1.2, on a flanking curve about an asymptote with reduced denominator n_0 only every n_0th voice attacks, and the attacking voice-sets cycle across successive flanking curves with a periodicity equal to n_0. Here only four voices are present, so attacks near asymptote $x = 1/3$ cycle through a repeating pattern in which Voices 20 and 23 both attack on a single flanking curve, followed by the other two voices attacking individually on successive curves. Finally, about $x = 1/2$ a duple meter emerges as attacks in odd and even voices, respectively, move toward alignment with one another.[29]

Example 9(a)–(d) shows array subsets with the total number of voices N' ranging from two to five. Asymptotes having denominators less than or equal to N' are shown in each case. Since higher-denominator asymptotes always possess some flanking curves without any sounding attacks on them, their associated meters are less likely to be heard.[30] Thus the number of distinctly perceptible metrical regions increases with increasing numbers of voices, while the sequence of relevant asymptotes in each case corresponds to Farey sequence $F_{N'}$ and the sequence of meters corresponds to the denominators thereof. Since no two consecutive Farey denominators are equal, successive regions always possess different meters.

At each metrical transition, either the intragroup time-intervals *on* the flanking curves about one asymptote grow to become intergroup time-intervals *between* attack groups on flanking curves about a new asymptote, or the reverse transformation occurs. Analyzing the relative sizes of these intervals and assuming that the total number of voices N' is small compared to the voice numbers involved, the Appendix shows that

1. such metrical transitions occur roughly when the current attack is fewer flanking curves away from the next Farey asymptote than from the previous one, and
2. the location in time of such a transition is roughly equal to the Farey mediant between the new asymptote and the old one, so that the extent of a metrical region corresponds to the Farey arc containing its asymptote.

These boundary locations are mere estimates. In practice, the precise perceived location of a metrical transition is influenced by the particular local details of the attack pattern, and may in fact be somewhat variable or subjective if the local attack-time intervals are large enough to render attack grouping ambiguous. Furthermore, the ear is typically reluctant to abandon a well-established metrical interpretation for a new one, a disposition that tends to delay such transitions (as illustrated by the brackets above Example 9[c]).

When the number of voices N' is not small compared to the voice numbers involved, the analysis becomes less straightforward. Among the complicating issues is the fact that intragroup attack-intervals increase in the lower region of a flanking curve, undermining attack-group cohesion. Another factor is that, in order for a metrical region to be perceptually established, the Farey arc delimiting it should be *at least* wide enough to accommodate one full period of the associated attack pattern. This period, however, is equal to the attack period of the highest voice ($1/n_{\max}$), while the bounds given for the Farey arc width in Section 1.1 are independent of $n_{\max}$, instead varying with the Farey denominator n_0 and the order of the Farey sequence (here N', the number of voices). Thus, as $n_{\max}$ decreases, higher-order metrical regions may fail to be perceptually established. Instead, they may be absorbed by neighboring metrical regions associated with lower-denominator asymptotes, fragment into various small rhythmic groupings, or (upon a first hearing) result only in a sense of metrical uncertainty.[31]

29 The foregoing alludes to five distinct hierarchical levels of perceptible rhythmic groupings. These are listed below in order from the most local to the most global:

1. individual attacks;
2. attack groups residing together on a single flanking curve, and having the character of *arpeggiando* figures or grace notes preceding or following a metrically stronger main attack;
3. a constant periodicity equal to that of the highest voice, which is variously subdivided into "tuplets" by the cyclical patterns of voices presented in the succession of attack groups;
4. the irregularly spaced transitions between these different n_0-tuple meters, which correspond to transitions between sets of flanking curves about asymptotes of different reduced denominators n_0;
5. the periodicity of the polyrhythm as a whole, which in Example 9 corresponds to unity.

If $n_0 = 1$, then levels 2 and 3 coincide as a single level, as at the opening of Example 9(c). Whenever n_0 equals the total number of voices then levels 1 and 2 coincide, since each attack group comprises only a single attack such as, for instance, about the asymptote $x = 1/4$ in Example 9(c). On the other hand, if a level-2 rhythmic grouping comprises four or more rhythmic units at level 1, then an additional level may appear between levels 1 and 2 if these units perceptually begin to group into twos and threes, as typically occurs. For the same reason, an additional level may sometime appear between levels 2 and 3.

30 Such metrical regions may sometimes be audible if they are wide enough to contain one or more complete metrical cycles. For instance, in Example 9(c), some listeners may detect a "5-feel" near $x = 1/5$ imparted by four attacks and a "rest." Similarly, in Example 9(a), listeners may be able to detect a "3-feel" near $x = 1/3$ comprising multiple 3-cycles with rests, and even a "4-feel" near $x = 1/4$ as well.

31 According to the Farey arc-width bounds, a full metrical period will definitely fit within the Farey arc in question whenever

$$n_{\max} > \frac{n_0(2N' - 1)}{2}.$$

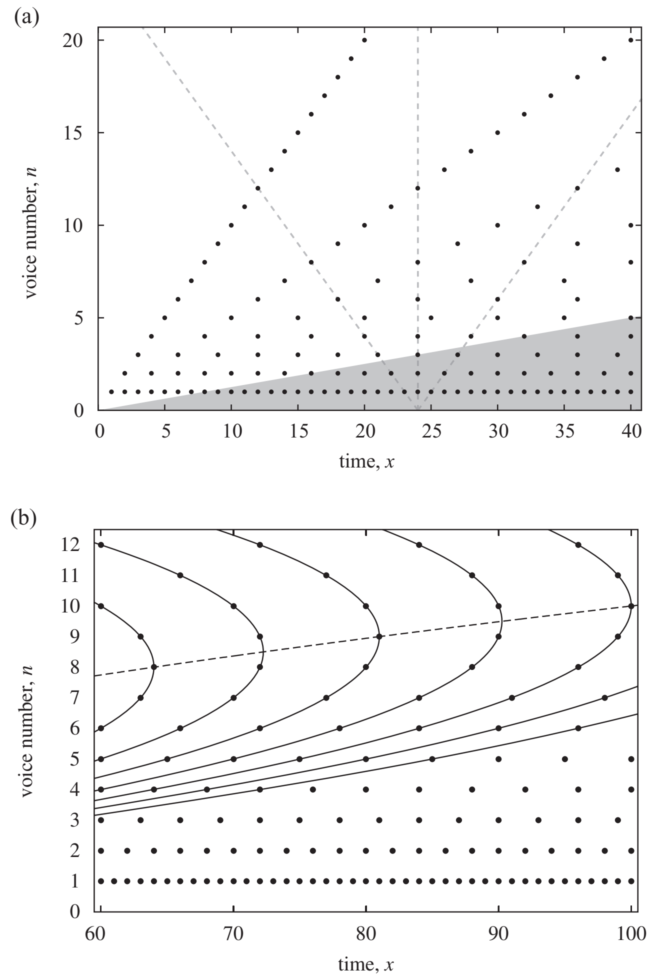
Such complications notwithstanding, Farey asymptotes and flanking curves remain a potential aid in the analysis, audiolization, and performance of low-voice-number polyrhythms by organizing their individual attacks into broader features. Example 9(e) shows that most attacks in an 8:7:6:5 polyrhythm can be regarded as falling on curves flanking $x = 0$. I hear this rhythm as being organized by the regular "beat" of the uppermost voice, relative to which the following features are scheduled: 1) an initial tutti attack, 2) two four-note decelerating arpeggiations (each beginning on the beat and descending stepwise across the four voices) with the second slower than the first and with a slight additional gap between them, 3) two notes straddling the last attack of the second arpeggiation, the first of which marks the top-voice beat, 4) a simultaneous dyad on the beat (not involving the last two voices heard), and commencing 5) a retrogression of all the preceding events.

2. multiplicative arrays

Unlike the canonical divisive array, the period of the canonical multiplicative array depends on the total number of voices present and increases without bound as this number increases. Individual voices, however, are periodic: Voice n attacks concurrently with every nth attack of Voice 1, which attacks at each positive integer value of the time variable x. Thus the time of the mth attack within Voice n is $x(m,n) = mn$, which constitutes the *defining equation* of the canonical multiplicative array.

Obviously, a voice in a multiplicative array attacks at time-value $x = mn$ if and only if its voice number n divides x; i.e., the voices attacking at any given time x are precisely those whose voice numbers divide x. Furthermore, successively higher voices enter at unit time-intervals; i.e., Voice n first attacks at time $x(1,n) = n$. Therefore, we may also say that voices attacking simultaneously are those whose voice numbers divide the voice number of the just-entered highest attacking voice, Voice x. When the voice number of this new voice is a prime, for instance, its entrance is accompanied only by Voice 1. These properties are illustrated in Example 10(a).

Potentially audible features of the multiplicative array include the straight lines described by the mth attack in each voice. These lines fan outward from the origin in Example 10(a), each ascending at a rate inversely proportional to m. Also visible are other radiating linear patterns around time-values that have relatively many divisors. In the example, one such pattern is emphasized using dashed lines about time value $x = 24$, which has divisors $\{1, 2, 3, 4, 6, 8, 12\}$.[32] The attacks delineating this fan occur at nearby x values that share one of these divisors.



example 10. *Two details from the canonical multiplicative polyrhythmic array: (a) Dashed lines mark voice* glissandi *about time-value* x = *24. Points within or on the border of the gray-filled region model the opening of Tenney's* Spectral CANON for CONLON Nancarrow. *(b) A partial family of attack parabolas, with the square-root curve indicated using a dashed line.*

Particularly clear *voice glissandi* traversing successive voices on successive attacks result whenever the focal x value has a sequence of at least a few successive integers among its divisors. For instance, this is the case for x values 12, 24, and 36, all of which share the divisors 1, 2, 3, and 4, as shown in Example 10(a). Such voice *glissandi* usually involve low voice numbers, since these appear most often as divisors.

Another feature of multiplicative arrays becomes increasingly visible at higher x values: families of leftward-opening *attack parabolas*, such as those illustrated in Example 10(b). The illustrated parabolas all have vertices residing on the curve $y = \sqrt{x}$, which is shown with a dashed line, but other families appear and coexist at greater values of x.[33] In Example 10(b), vertices occur

For $n_0 = N' = 5$, this inequality is $n_{\max} > 22.5$. In practice, however, even this bound may not be sufficient to ensure that a definite meter is perceived—on an initial hearing two metrical downbeats or even two full metrical periods may be necessary rather than a mere single period. Thus, in the polyrhythm of Example 9(d) the fleeting quintuple pattern near $x = 2/5$ can easily be subsumed by the following duple pattern.

32 Like the one about $x = 0$, this fan-pattern actually has many radiating blades, but only the three audibly prominent ones are marked on the figure.

33 Ventrella (n. d.) offers visualizations, modeling, and discussions of these and other patterns among the integers.

at times x that are alternately perfect squares (such as 64, 81, and 100) or near pronic numbers (which are products of two successive integers, such as $8 \times 9 = 72$ and $9 \times 10 = 90$). With the exception of single attacks falling on a vertex, attacks are paired in simultaneous dyads, one on each arm of a parabola, with the average of their voice numbers always equaling the height of the vertex. Because the time-intervals between dyads on a given parabola increase (linearly) away from the vertex, attacks on other nearby parabolas typically intervene between them. This often makes individual parabolic arcs easier to see than to hear, unless they are differentiated via some attribute other than pitch alone (such as timbre). However, near a vertex the intervallic convergence of dyads toward unisons or consecutive-voice dyads may sometimes be heard. Example 10(b) also shows *voice glissandi* in the lower voices, with a particularly prominent instance appearing about time-value $x = 60$.

Multiplicative arrays can be used to model a familiar (non-rhythmic) acoustical structure. When a "harmonic series" is sounded by distinct instruments assigned to successive partials, it actually comprises a collection of complex tones, each of which has its own harmonic spectrum; in other words, it is a collection of individual harmonic series whose fundamental frequencies also fall in a harmonic series. Suppose that individual complex tones are indexed from lowest to highest by n. If frequency units are chosen such that the lowest tone has unit fundamental frequency, then the ensemble of sounding partials can be modeled by a multiplicative array wherein each "voice" corresponds to a single complex tone whose individual partials are indexed by m and have frequencies $x = mn$. In other words, the individual complex tones correspond to the voices of Example 10(a), with each individual harmonic series increasing in register from left to right.

Multiplicative arrays feature in two compositions by Tenney: *Three Harmonic Studies*, III and the first part of *Spectral CANON for CONLON Nancarrow*. I have analyzed the former elsewhere,[34] and the latter is examined below.

3. *SPECTRAL CANON FOR CONLON NANCARROW* (1974)

A classic example of process music, James Tenney's *Spectral CANON for CONLON Nancarrow* is one of the most distinctive and arresting works in his large and varied compositional output. Its design was begun years before the first realization and was refined on a teletype terminal. Nancarrow himself punched the final piano roll on his own roll-punching machine. Realization was achieved in 1974 with assistance from composer Gordon Mumma, using a player piano in Santa Cruz, California. Commercial recordings of this realization are available, and a reference score and even a reproduction of the piano roll have also been published.[35]

The work is a canon in twenty-four voices lasting roughly three and a half minutes. Each voice of the canon is assigned a different single pitch, which it repeats undampened again and again (as shown in Example 3). The pitches used are the first twenty-four harmonics of A_1 (55 Hz), to which the required piano strings are precisely retuned. The sequence of inter-attack durations is identical in all voices, although the higher a voice is pitched the later it enters—hence the canonic aspect. The duration sequence is monotonically decreasing at first, but once a voice has sounded a specified number of attacks (a number that is the same for all voices), the sequence begins to retrograde. Only the lowest-pitched voice finishes its retrograde, the piece ending at the moment when this happens. Clearly these pitch and temporal resources are narrowly restricted in themselves, but their *interaction* produces complex and surprising results.

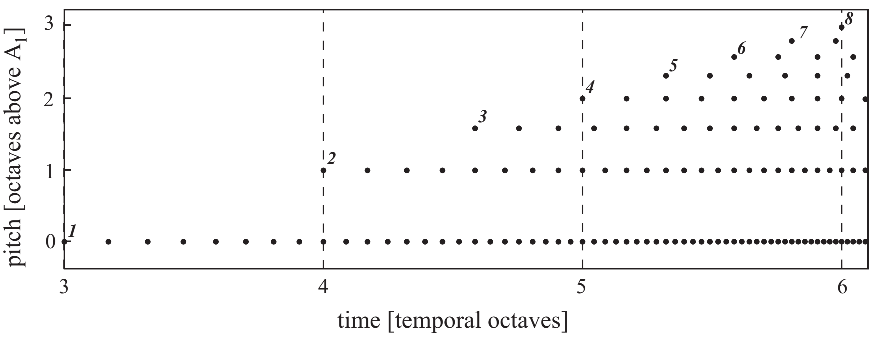
The piece opens with A_1 slowly repeated, its constituent harmonics ringing above the low fundamental. As successive voices enter, gradually ascending the harmonic series, polyrhythms emerge, which increase in complexity until the combination of many chiming voices produces a welter of sound. The strong beat initially supplied by the lowest voice gradually becomes a steady drone as the repetition rate of its undampened tone increases and the sense of unifying pulse temporarily disappears. Slowly, however, a new and striking variety of order creeps into the lower voices, as rising and falling *glissandi* sweep progressively higher along the harmonic series, punctuated by simultaneous attacks in multiple voices. These *glissandi* subsume successively higher voices until, as they reach the highest, the piece concludes dramatically with all twenty-four voices sounding simultaneously for the first time. Due to perfect harmonic fusion, the result is surprisingly perceived less as a chord than as a single tone pitched at A_1, which was, of course, the first pitch heard at the opening. It is as though the constituents of this single complex tone appear torn asunder in the core of the piece, but are powerfully welded together again in its concluding gesture.

There exists an unreleased longer version (5′25″) of Tenney's *Spectral CANON* dating from 1990, extended and realized by composer Clarence Barlow using a computer-driven player piano while he and Tenney were both professors at the Summer Courses for New Music in Darmstadt, Germany. It continues beyond the total simultaneity at the end of the original version, allowing all voices to finish their retrogrades with voices falling silent one by one in the order in which they first entered. Further high-register *glissandi* and simultaneities are heard among the remaining voices as they play out, the thinning textural density allowing surprising melodies to gradually emerge. For reasons of analytical convenience, the following study will address this extended version of the piece (the conclusions will straightforwardly apply to the shorter original version as well).

An analytical starting point can be found by observing that the lowest voice in the graphic score of Example 3(a) qualitatively

34 See Wannamaker (2008, 102–5).

35 Recordings include *Cold Blue* ([1984] 2000), and *Donaueschingen Musiktage 1994* (1995), as well as the audio cassette accompanying Tenney (1984). See Example 3.online for an audio recording. The reference score is published in Tenney (1976) and the reproduction of the piano roll is published in Tenney (1984). Polansky (1983, 223–25) contains an analysis of the work.



Example 11. *Graphic score for the opening of* Spectral CANON for CONLON Nancarrow. *Time is measured in units of temporal octaves relative to the temporal fundamental of Voice 1. Voices are numbered in italics at their entrances.*

resembles a harmonic series turned on its side, its inter-attack durations decreasing from left to right just as the pitch intervals between successive harmonics diminish in size as the series ascends (viewed in log-frequency space). Indeed, by the composer's own account, composition of the work involved experiments with various juxtapositions of such "durational harmonic series."[36] By analogy with a harmonic series of pitches, it is thus possible to speak of temporal intervals (i.e., durations) such as temporal octaves within each canonic voice. In particular, by direct measurement it can be confirmed that this durational harmonic series is missing its lowest seven "harmonics" (i.e., attacks), so that it begins in its "fourth octave." Thus, as Polansky points out,[37] the first eight sounding attacks occur during a temporal octave of roughly 23.5 seconds, the following sixteen attacks over the course of a similar interval, the following thirty-two attacks similarly, and so forth. This is illustrated by Example 11, which provides an expanded graphic score for the opening of the work.

Voice-entrance times also articulate a durational harmonic series, with the same octave duration. In this case, however, the series begins in its first octave, wherein a single new voice appears, while two voices enter in the second octave, and so forth, as shown in Example 11. Note that, despite the exponentially increasing rate of voice entrances, the pitch of entering voices ascends linearly in time due to the logarithmic narrowing of pitch-intervals within the harmonic series.

The compositional features described above are summarized in the following four conditions, which fully specify the pitch-time structure of *Spectral CANON for CONLON Nancarrow*.

1. Numbering the voices from 1 to $N = 24$ beginning with the lowest, the pitch of Voice n corresponds to the nth harmonic relative to a fundamental of A_1.
2. The rhythm in all voices is identical, apart from the canonic delays, and displays retrograde symmetry, with Voice 1 beginning its retrograde at the moment when Voice N enters.
3. Just as in a harmonic pitch series, wherein the pitch interval from the fundamental to a particular harmonic is given by the logarithm of its harmonic number, the sounding attacks of the forward portion of the attack-time sequence in Voice 1 occur at times

$$\{\log_2 8, \log_2 9, \log_2 10, \ldots\}.$$

The time scale employed here uses units of temporal octaves (not seconds) and a temporal origin located three temporal octaves before the first attack of the piece.[38] The temporal octave c is a constant chosen such that the time-interval between the first two sounding attacks is four seconds; i.e.,

$$c = \frac{4}{\log_2(9/8)} \approx 23.5 \text{ seconds.}$$

4. The entrance of Voice n is delayed relative to that of Voice 1 by a time-interval of $\log_2 n$ temporal octaves.[39]

In each voice, the initial seven attacks are presumably omitted from the durational harmonic series on aesthetic grounds. Note that, if they were retained, the time-interval between the first and second attacks would be a full temporal octave of $c \approx 23.5$ sec. The following analysis is simplified by treating these missing attacks as present but non-sounding. That is, I will locate my temporal origin at the "fundamental" of the complete forward durational harmonic series in Voice 1, but the reader should bear in mind that in each voice the first seven harmonics of its temporal fundamental (i.e., the first seven attacks) are *notional*, as

36 Tenney (2003).

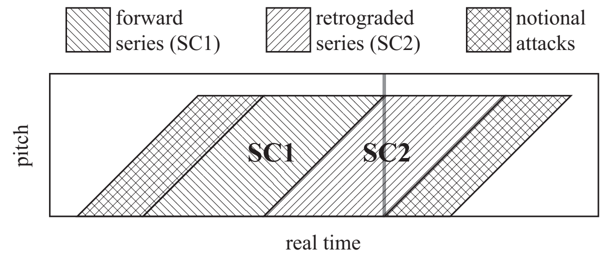
37 Polansky (1983, 223).

38 All logarithms appearing in this article have base 2. Therefore, the unit of pitch (or, here, duration) is always an "octave," since $\log_2(2/1) = 1$. In other words, the unit of pitch corresponds to a ratio of 2:1 in frequency.

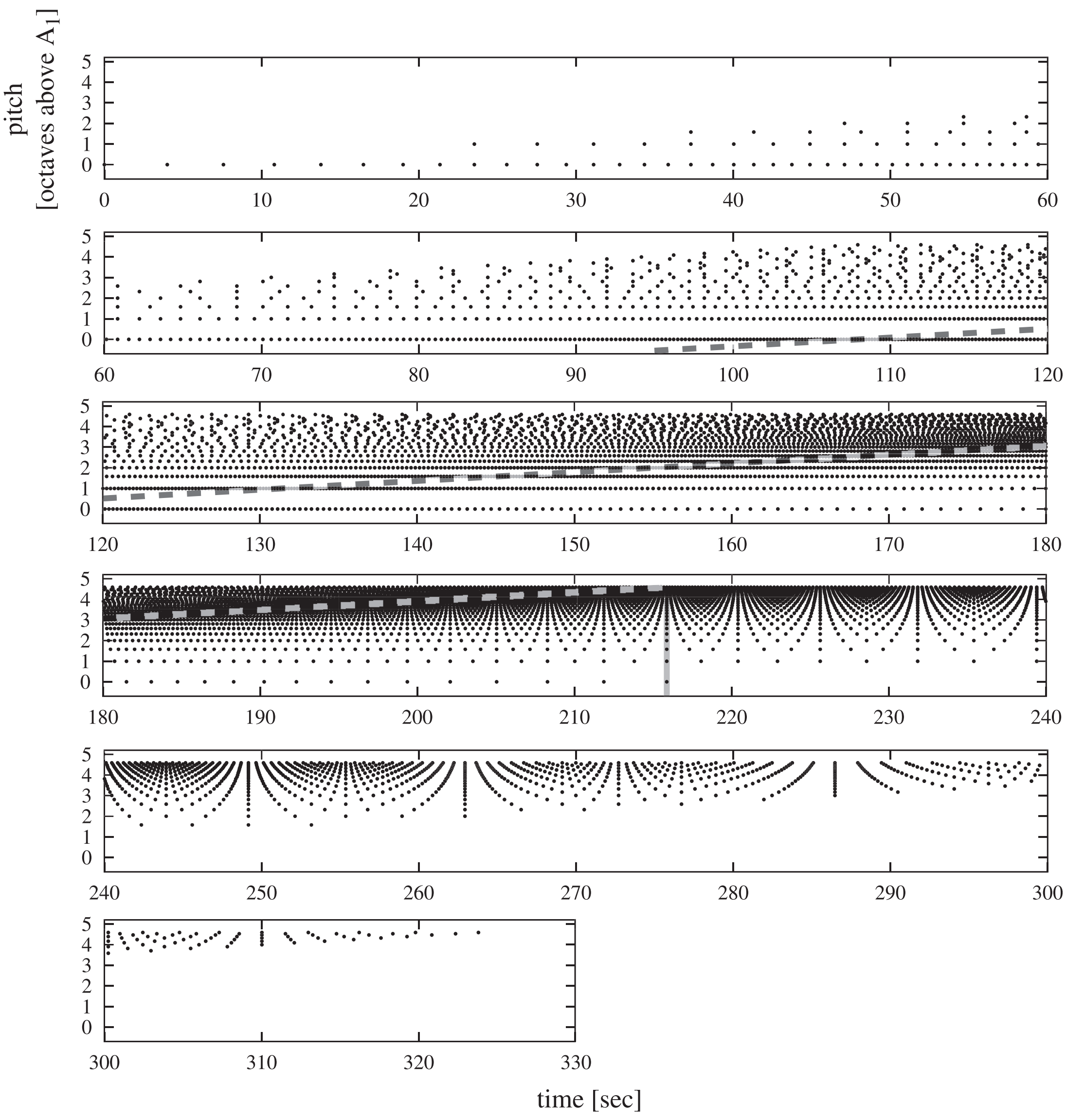
39 Possible variations on these specifications suggest themselves, such as reversing the order of voice entrances or beginning in each voice with the "retrograded" series of decreasing durations instead of the "forward" series of increasing durations. In fact, Tenney composed three such *Spectral Variations* (1991/1998), realized by composer Ciarán Maher and premiered in 2007. These variations can be heard online (Maher [n. d.]).

opposed to *sounding*. The same caveat applies to the last seven attacks in each retrograde.

My analysis will initially ignore that the collections of forward and retrograded attack-time sequences overlap in time, instead treating them separately and referring to the forward collection as SC1 and the retrograde as SC2. The overall form of the work is diagrammed in Example 12, where the gray vertical line represents the conclusion of Tenney's original realization. A complete graphic score of the extended realization is given in Example 13 as algorithmically generated by the author from the foregoing specifications.



EXAMPLE 12. *Overall form of* Spectral CANON for CONLON Nancarrow *in Barlow's extended 1990 version. The gray vertical line marks the conclusion of Tenney's original 1974 version.*



EXAMPLE 13. *Complete graphic score for* Spectral CANON for CONLON Nancarrow *in Barlow's extended 1990 version as algorithmically generated from the compositional specifications. The vertical gray line marks the conclusion of Tenney's original 1974 version. Attacks preceding the oblique dashed line are in SC1 (i.e., are presenting forward durational series), while those following it are in SC2 (i.e., are presenting retrograded durational series). (See* MTS *online to listen to the complete recording of Barlow's extended version.)*

3.1. ANALYSIS OF SC1

Locating the temporal origin at the first notional attack in Voice 1 and adding the initial time-offset for Voice n, we obtain the following expression for the time $x(m,n)$ of the mth attack in Voice n in units of temporal octaves.[40]

$$x(m,n) = \log_2 n + \log_2 m = \log_2 mn \qquad (1)$$

The graphic score excerpt in Example 11, which depicts the beginning of SC1, shows a clearly visible structural correspondence to the canonical multiplicative array shown in Example 10, especially if the first seven notional attacks in each voice are excluded (i.e., those attacks outside of the shaded region in Example 10). The only differences apparent in the graphic score are the gradual *accelerandi* within each voice, and the similar gradual narrowing of the pitch interval between higher voices (reflecting the progressive narrowing of intervals within the harmonic pitch series). Informally, one could remove these "distortions" from Example 11, thus yielding the canonical multiplicative array, by imagining the example's visual space as an elastic sheet and independently and non-uniformly stretching its vertical and horizontal dimensions, respectively, until all of the voices become evenly spaced and the *accelerandi* disappear. Formally, we can apply the independent *coordinate mappings* $x \rightarrow 2^x$ and $y \rightarrow 2^y$. These map each point (x, y) onto a point $(2^x, 2^y)$, thus imposing an exponential stretching that eliminates the original logarithmic variation in each dimension.[41] In particular, this eliminates the logarithmic function from the right-hand side of Equation (1) so that it becomes $x(m,n) = mn$, the defining attack-time equation of the canonical multiplicative array shown in Example 10.

Since the mapping of each coordinate is continuous and independent of the other, these mappings will change the order of neither the horizontal nor the vertical coordinates of any pair of attacks in the array. In particular, observations regarding attack multiplicities, orders, and coincidences in the canonical multiplicative array will thus also apply to SC1. The remainder of this section will treat SC1 as a canonical multiplicative array, with the understanding that this array is subjected to a gradual *accelerando* in the actual music. In other words, x below refers to the transformed (exponentiated) time-coordinate, not a real (clock) time-coordinate. In particular, it should be noted that the voices of the canonical array are not in exact rhythmic canon but in augmentation canon (since later voices enter at slower tempi), but that the aforementioned *accelerando* engenders the exact canons of the actual music.

As indicated above, in each voice the attacks numbered $m = 1$ through $m = 7$ are notional (non-sounding). Let M denote the total number of attacks in the forward version of the duration sequence, including the seven notional ones. Then, following the compositional specifications enumerated above, if the total duration of Voice 1's forward sequence is equated with the time of the first *sounding* attack in Voice $N = 24$, we obtain $x(M,1) = x(8, N)$ or

$$M = 8N = 8 \times 24 = 192.$$

Therefore, the number of attacks in a complete statement of the forward durational sequence is 192 but, since the first seven are notional, only 185 actually sound. The total number of sounding attacks in Voice 1 is $2 \times 185 - 1 = 369$, since the forward and retrograde presentations share one attack.[42]

As shown in Example 11, the polyrhythms of the opening appear in the sequence {2:1, 3:2:1, 4:3:2:1, ...} as successive voices enter. The first sound in the piece is the entrance of Voice 1 with eight attacks of A_1 at $x = 8, 9, 10, \ldots, 15$. Voice 2 enters at $x = 16$ with A_2 reiterated on every second attack of Voice 1, followed by Voice 3 at $x = 24$ with E_3 on every third attack of Voice 1, and so forth. At this point the period of the total polyrhythm articulated by Voices 1 through 3 is the *Least Common Multiple* (LCM) of their voice numbers, LCM{1, 2, 3} = 6, but with the entry of Voice 4 this period becomes 12. This exceeds the eight-unit time-delay between voice entries so that a complete polyrhythmic period is not presented before a new voice enters. Indeed, LCM{3, 4} = 12, so that even the period of the upper two voices alone fails to complete before a new voice entrance. In order to discern polyrhythmic cycles after the entrance of Voice 4, it is necessary to restrict attention to just a few voices, a task that becomes more challenging as new voices enter on pitches that are increasingly close to one another.

As is the case with any multiplicative array, voices heard attacking simultaneously in SC1 are those whose voice numbers divide the time value x (as discussed in Section 2), but with the further restriction that these attacks must be sounding rather than notional. An attack in Voice n at time x is sounding if and only if $n \leq N = 24$ and its attack number $m \geq 8$. The imposition of the last constraint effectively deletes from Example 10(a) all attacks falling outside of the gray-filled region. In contrast with the straightforward linear ascent visible in Example 10(a), this yields a sequence of highest-sounding voices that is irregular, since the proper divisors of successive integers are not simply related to one another.

During the first two minutes of the recording, *voice glissandi* of the sort described in Section 2 are audible up and down the low members of the harmonic series, although these become less apparent as the ear is drawn to higher entering voices and the

40 Here I invoke the summation identity for logarithms, $\log a + \log b = \log ab$.

41 The mapping of the vertical coordinate corresponds to changing from a pitch to a frequency scale.

42 The reference score (Tenney [1976]) contains some inaccuracies in its latter half, particularly with respect to the note counts in each voice. For instance, the number of notes in the forward-durational series in Voice 1 (i.e., up to the entrance of Voice 24) is the expected 185, but the number in its ensuing retrogression differs, and both tallies differ from the total number of notes in Voice 24. All of these counts should be equal according to the compositional specifications for the work. Furthermore, in the original player-piano roll (Tenney [1984]) all of these note counts agree with one another and with the values computed herein.

tempi in the lower voices increase.[43] Between eighty and 100 seconds into the recording, simultaneous dyads describing *attack parabolas* among the upper voices can be heard converging to unisons or consecutive-voice dyads, the latter close intervals being particularly salient features of the texture. Both types of structure are visible in the score excerpt of Example 14, where gray lines have been added to emphasize them.[44] During the *Spectral CANON*'s rapid and complex middle portion, the irregular sequence of highest pitches engenders fleeting melodic lines, which are sometimes compound. Typically, a perceived melodic snippet comprises attacks among high voices that are close to one another in time and pitch and perceptually segregated from adjacent snippets by a temporal gap. Such gaps appear at time-values x that have no large divisors less than the highest sounding voice (e.g., at prime values of x, when only Voice 1 sounds). Musical interest is greatly enriched by the ephemeral appearance, coexistence, interaction, and disappearance of these diverse audible structures.

3.2. ANALYSIS OF SC2

Example 3(b) visibly resembles the divisive array structure of Example 4(b). Differences include the obvious general *ritardando* and the familiar progressive narrowing of harmonic-series intervals between voices, both of which can be eliminated via suitable coordinate mappings (as in the analysis of SC1). As indicated above, this analysis treats Barlow's extended realization of *Spectral CANON*, in which all voices are allowed to finish their retrogrades, and regards Tenney's original realization, wherein all higher voices are truncated when Voice 1 finishes, as a special case (see Example 13). Finally, recall that, for analytical convenience, the model includes the seven *notional* attacks in each voice at the work's beginning and also in its concluding retrogressions.

I will begin by choosing a new time origin located at the final notional attack in Voice 1 (i.e., at the image under retrogression of its first notional attack). Since the entrance of Voice n was delayed by a time-interval of $\log_2 n$ with respect to Voice 1, its conclusion will be similarly delayed. Thus, in Voice n, the (real) time-coordinate of the mth attack, *counting backward in time from the final notional attack in that voice*, is[45]

$$x(m,n) = \log_2 n - \log_2 m = -\log_2 \frac{m}{n}. \qquad (2)$$

Applying the coordinate mapping $x \to 2^{-x}$ yields

$$x(m,n) = \frac{m}{n},$$

which is the defining equation for the canonical divisive array shown in Example 4. The role of this mapping perhaps becomes clearer if it is decomposed into two stages: $x \to -x$ followed by $x \to 2^x$. Respectively, these are a reflection about a vertical axis (i.e., a time-reversal) and an exponential "stretching" that eliminates the logarithm. The application of these two transformations to the horizontal dimension of SC2, plus an independent $y \to 2^y$ stretching transformation applied to its vertical dimension, converts it into the canonical divisive array. The reader may wish to compare Examples 3(b) and 4(b), keeping in mind that all voices in the former should be extended so that their retrogressions become complete. The remainder of this section will treat SC2 as a canonical divisive array, with the understanding that this is subjected to a gradual *ritardando* in the actual music. In other words, as in the analysis of SC1 above, x will refer to the transformed (exponentiated) time-coordinate, not a real (clock) time-coordinate.

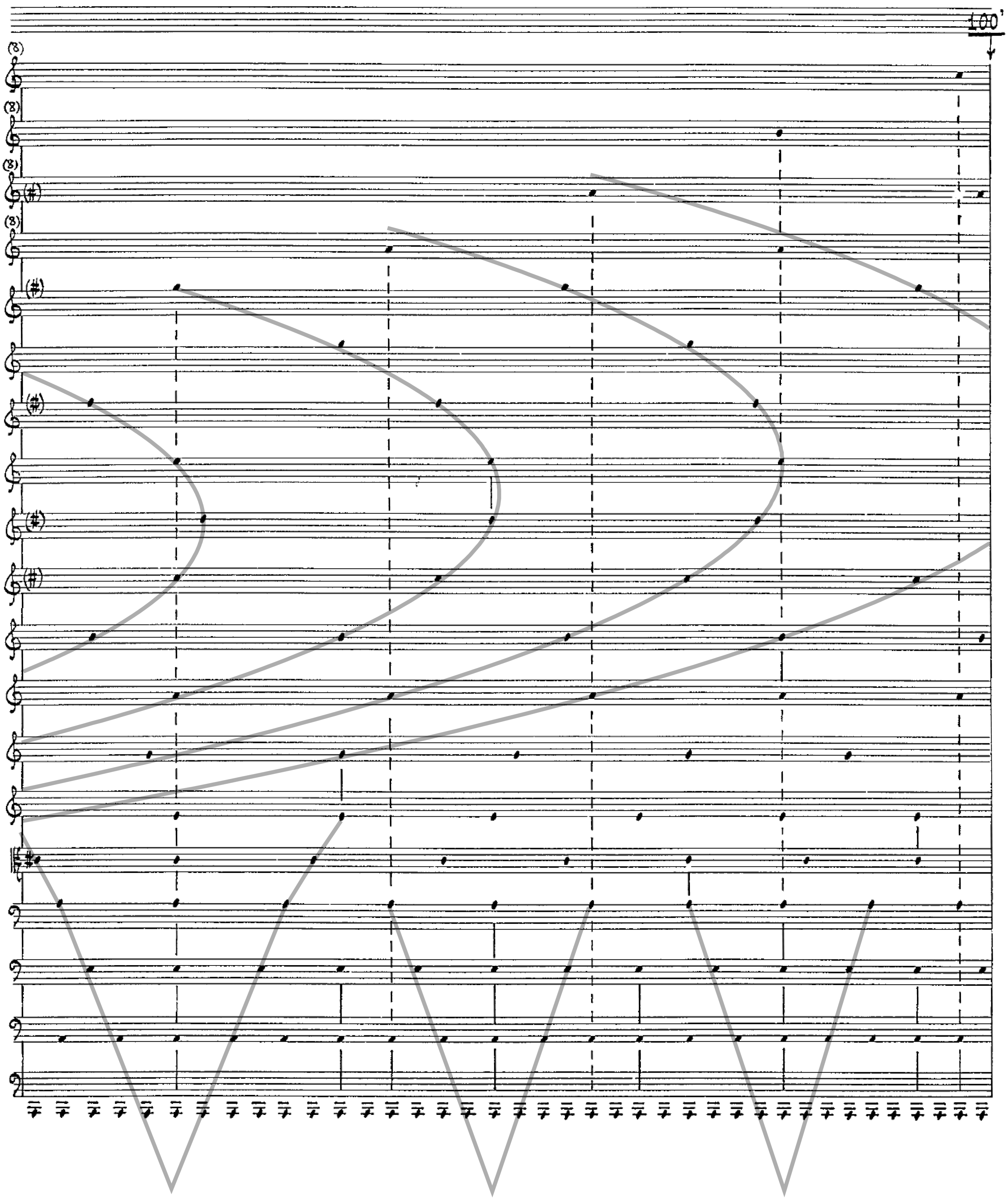
As is always the case in a divisive array, when Voice 1 attacks, all other voices attack simultaneously with it. The last sounding attack in Voice 1 has attack number $m = 8$, and it is with this attack that Tenney's original realization of the *Spectral CANON* ends. Earlier simultaneities between voices in SC2 are clearly audible, but do not involve all twenty-four voices, since some high ones are still presenting forward forms of the durational sequence (i.e., they belong to SC1). The extended version of the piece continues and exhibits simultaneities among those upper voices still sounding, all of which belong to SC2 since, by that point in time, they are all presenting retrograde forms of the duration sequence.

Other structural features of the divisive array are clearly audible in the texture of the original version as its conclusion nears, and are visible in the graphic score of Example 15. These include the relatively large attack-free Farey buffer zones around integer values of x. The striking harmonic *glissandi* noted earlier are associated with flanking curves. The most salient are those about integer values of x and with indices $j = \pm 1$, since these precede or follow the aforementioned buffer zones. My ear usually follows the descending curve immediately following such a buffer zone, but due to the pervasive sharing of attacks between *glissandi* it is often lured by an ascending curve that is not the last before the next buffer. Whichever curve my ear follows, others provide pre- or post-echoes. Although less salient, *glissandi* about half-integer (and even third-integer) values of x are nonetheless audible, especially if attention is directed to the high register where these *glissandi* are steep and abut their own small Farey buffer zones. The lowest-voice profile of the array is audible, especially those attacks involving the lowest five voices. In intermediate registers the complex chatter of divisive polyrhythms is also discernible, like a chaotic rainfall of harmonics out of which the broader audible figures emerge. As Barlow's extended version continues beyond Tenney's original conclusion, lower voices begin to fall silent as they conclude their

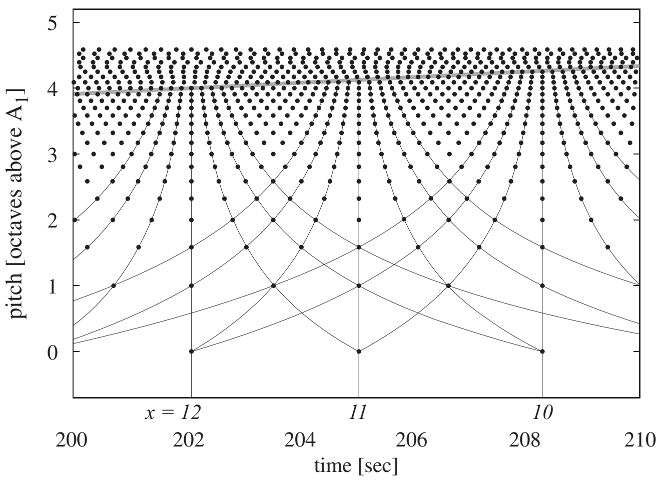
43 They remain visible in the score, however, where they sometimes extend into high voice numbers to create fan-like patterns (see Example 13: 120–180 seconds).

44 The alignment between curves and attacks is approximate due to changes of clef in the score and the non-linear mapping of pitch onto musical staves. Some voice *glissando* lines are bent for the same reasons.

45 The logarithm identities $\log(a) - \log(b) = \log(a/b)$ and $-\log(\mathrm{b}) = \log(1/b)$ are used here.



EXAMPLE 14. Spectral CANON for CONLON Nancarrow, *excerpt from the reference score (Tenney [1976], 90–100 seconds). The composer marked the entrance of new voices with dashed vertical lines, and simultaneous attacks in adjacent voices with solid vertical lines. In this figure, additional solid gray lines indicate selected attack parabolas and voice* glissandi. *Copyright 1974 by Smith Publications, Sharon, VT. All rights reserved. Used by permission.*



EXAMPLE 15. *Graphic score of* Spectral CANON for CONLON Nancarrow *near the conclusion of Tenney's original 1974 version. Voices above the thick gray line are still part of SC1 (i.e., they have not yet begun to retrograde, and thus are still increasing in tempo). Total simultaneities among the lower voices are indicated with thin lines, as are segments of selected flanking curves about them. The horizontal axis is linear in real elapsed time, the values of which are indicated in roman type, while values of* x, *the transformed time-variable for SC2, are in italics.*

retrogrades. This more clearly exposes the high-register tails of the *glissandi* along flanking curves, as shown in Example 13. Not only do curves of distinct *j* value become more easily discernable about integer-valued asymptotes, but so do whole families of such curves about asymptotes with higher denominators such as $n_0 = 2$, 3, and 4. In the closing seconds of the music, there are even suggestions of the sort of metrical patterns explored in Section 1.4, although these are rendered ephemeral by the ongoing loss of voices.

3.3. FORMAL DESIGN

Over the course of SC1, the music undergoes a gradual progression from stark simplicity to chaotic complexity via concurrent increases in tempo, registral compass, number of sounding voices, polyrhythmic complexity, and harmonic complexity. In SC2, sensible order is progressively recovered via a different process. Compared to the complex welter of the work's middle portion, the texture seems to simplify as the music draws to a close. This is a product in part of the decreasing reiteration rates of the lower voices, which determine the sensed tempo of the music by demarcating the audible durations that are subdivided by higher-pitched attacks. Also, the *glissandi* and simultaneities emerge as gestalts subsuming many individual attacks so that an audibly complex texture comprising a counterpoint of many brief, rapid, and unpredictable melodic snippets is gradually superseded by a texture comprising larger composite features. The perception of an inexorable process of increasing complexity and fragmentation gradually yielding to a grand order subsuming all musical features constitutes the work's large-scale formal and dramatic arc.

The progressive emergence of this new order is effected by the structural "splicing" of SC1 and SC2 that occurs in the middle of the piece, and which is represented in Example 12. The smoothness of the transition is promoted by adherence to a static pitch reservoir, and by the very gradual tempo change within each voice, which ensures rhythmic continuity when its retrogression begins. What is perhaps surprising, however, is that simple intravoice retrogression (coupled, of course, with a specific set of canonic lags) produces the formally crucial transition between a multiplicative and a divisive polyrhythmic array. What takes place is clearly subtler than just a structural "crossfade" between the two array types conducted under the cover of a rhythmically chaotic texture.

The analysis above has skirted the gradual logarithm-induced *accelerandi* and *ritardandi* of the durational sequences in *Spectral CANON* by introducing coordinate mappings that remove those tempo changes while preserving attack orderings and coincidences. This permitted conclusions about the music's structure to be drawn from the earlier analyses of canonical array structures. It is in the work's crucial transition from a multiplicative to a divisive array structure, however, that the logarithms play an essential role.

Table 1 compares the defining attack-time formulae for the canonical arrays versus their logarithmic images—Equations (1) and (2) above—that appear in *Spectral CANON*.[46] The logarithmic mapping decomposes the product $x(m,n) = mn$ into a sum of logarithms, and the quotient $x(m,n) = m/n$ into a difference thereof. This is achieved by virtue of the additive and subtractive logarithmic identities: $\log ab = \log a + \log b$ and $\log a/b = \log a - \log b$. Logarithmic functions are unique in thus relating products and quotients to sums and differences. Accordingly, no other transformation of the canonical attack-time equations will decompose them into an intervoice time-delay term dependent only on voice number (*viz.* $\log_2 n$) and an added or subtracted intravoice attack-time term dependent only on attack number (*viz.* $\log_2 m$).[47] The positive intervoice canonic delay $\log_2 n$

46 Before proceeding, two minor (but potentially confusing) issues regarding the entries in this table should be addressed. First, as previously indicated, the logarithmic equations describing SC1 and SC2 assume different time origins (located at the first and last notional attacks in Voice 1, respectively). This fact, however, does not affect the internal structures of these arrays. Second, in the table's lower-right (SC2) equation, the appearance of the negative sign preceding $\log_2(m/n)$ indicates that not only has the attack-time structure been subjected to a logarithmic mapping relative to the canonical version above it, but also that the entire resulting array has then been retrograded in time. In other words, this retrogression is distinct from the intravoice retrogressions, being a wholesale reflection of the array about a vertical axis. This simple reversal, of course, does not essentially alter the character of the divisive array either.

47 For a contrasting example, note that in the canonical arrays themselves the inter-attack durational sequences are n-dependent (i.e., different in each voice). It is the logarithmic transformation that turns these arrays into literal rhythmic canons.

TABLE 1. *The defining equations of the canonical multiplicative and divisive arrays, compared with the logarithmic images thereof that model SC1 and SC2.*

	multiplicative (SC1)	divisive (SC2)
canonical:	$x(m,n) = mn$	$x(m,n) = \frac{m}{n}$
logarithmic:	$x(m,n) = \log_2 mn = \log_2 n + \log_2 m$	$x(m,n) = -\log_2 \frac{m}{n} = \log_2 n - \log_2 m$

appears in the attack-time equation for both SC1 and SC2, while the intravoice sequence $\log_2 m$ appears as an addend in the first and a subtrahend (i.e., retrograded) in the second. Thus it is the specifically logarithmic variation in the intravoice and intervoice attack-time sequences (inherited from the composer's choice of the harmonic pitch series as his rhythmic model) that induces the work's pivotal formal transformation from a multiplicative to a divisive polyrhythmic array via simple intravoice retrogression.

3.4. POSTSCRIPT: COORDINATION OF PITCH AND RHYTHMIC STRUCTURES

Cowell's motivating analogy between frequency ratios in the harmonic series and tempo ratios in divisive polyrhythms may have proven compositionally fertile, but it is not philosophically unproblematic. It might be taken to suggest that pitch intervals and rational polyrhythmic relationships are fundamentally similar because both are essentially determined by frequency ratios, differing only in the absolute rates of repetition involved.[48] This straightforward *physical* similarity does not, however, translate into a correspondingly straightforward *psychological* one, because pitches and rhythms are starkly different sorts of percepts.

Still, the possibility that Cowell's analogy might have a psychological counterpart should not be dismissed offhandedly. The uncontroversial fact that no one is likely to confuse a pitch interval with a polyrhythm does not entail that none of the products of rhythm and pitch perception bear comparison. These are multifaceted faculties, as becomes clear whenever the ear is confronted with multiple discriminable pitches or tempi whose relationships become objects of perception in addition to their individual attributes. Thus it is worth briefly considering what psychological significance frequency and tempo ratios *do* possess.

On the one hand, consider an arbitrary polyrhythm comprising two voices in a tempo ratio $n_1\!:n_2$. Such a polyrhythm is periodic in time with a period equal to the Least Common Multiple of n_1 and n_2. If the tempo ratio is reduced, then $\mathrm{LCM}(n_1,n_2) = n_1 n_2$.[49] This product corresponds to the number of even subdivisions of the total period required in order to "rationalize" the polyrhythm such that all attacks fall on a subdivision boundary. It thus provides a relative measure of aural and performance complexity.[50] It predicts, for instance, that the polyrhythms 3:4 and 6:8 have an identical complexity measure of 12—since both reduce to 3:4—but that each is simpler than 3:5, whose complexity measure is 15.

On the other hand, consider two complex tones with rationally related fundamental frequencies n_1 and n_2. Their partials will coincide in frequency at $\mathrm{LCM}(n_1,n_2)$ and multiples thereof. In post-Helmholtzian theories of consonance, the prevalence of such coincidences correlates with sensory consonance.[51] Thus (again assuming that $n_1\!:\!n_2$ is reduced) a low value of $\mathrm{LCM}(n_1,n_2) = n_1 n_2$ will be associated with a relatively consonant or "harmonically simple" dyad, while a higher value will be associated with a relatively dissonant or "harmonically complex" dyad.[52] This measure predicts, for instance, that the frequency ratios 3:4 and 6:8 have an identical harmonic complexity of 12 and that each is simpler than 3:5, the complexity of which is 15.

The above complexity measures for polyrhythms and harmonic intervals are formally identical. While rhythmic and harmonic complexities remain qualitatively distinct, each correlates with difficulty in identification, audiolization, and performance. In this limited sense, Cowell's analogy between physical frequency and tempo ratios has an inherent psychological counterpart.

48 However, it should be noted that the initial phases of the participating cycles affect a polyrhythm, but not a pitch interval.

49 For a discussion of this and other properties of Least Common Multiples, see Shockley (1967).

50 This assumes that the tempo of the subdivisions has a manageable value.

51 See Plomp and Levelt (1965, 555–56, 560); Roederer (1995, 165–68). The senses of "consonance" and "dissonance" intended here are sensory rather than functional, and "harmony" denotes only the sensory relationship among members of an unordered tone-set. It is assumed that comparisons are made using a fixed timbre and register, since otherwise the amplitude of participating partials and variation in critical bandwidth will contribute to the assessment of sensory consonance and dissonance; see Roederer (1995, 168).

52 $n_1 n_2$ appears as a measure of harmonic complexity for dyads as early as ca. 1563 in the writings of Giovanni Benedetti (Palisca [1961, 108–9]). It also appears in the psychoacoustical literature (Vos and van Vianen [1985, 180]) and features prominently in Tenney's theoretical writings on harmony in the form $\log_2(n_1 n_2)$ (Tenney [1993, 153]).

Whatever musical significance this correspondence attains will, however, depend on the specific use that is made of it.

Tenney's *Spectral CANON* provides an example of a structural analogy between pitch and duration that induces definite audible correspondences between harmonic and polyrhythmic complexities. Rhythmically, the opening comprises a collection of harmonic durational series whose fundamentals also reside in a harmonic durational series. One might describe the whole as "harmonic series on a harmonic series" for short. Note the analogy between this temporal structure and the pitch structure it is unfolding. The discrete piano tones are themselves harmonic complexes whose partials are arrayed in a harmonic pitch series, while their fundamental pitches also reside in a harmonic pitch series. That is, they also constitute "harmonic series on a harmonic series," albeit in the domain of pitch rather than time. More formally, the rhythmic structure of SC1 can be modeled as a canonical multiplicative array (by means of the coordinate transformations previously described). By compositional construction, the durational sequence in Voice n has "fundamental time" n, so that its mth attack occurs at time $x(m,n) = mn$. However, if $A_1 = 55$ Hz is adopted as the unit of frequency, then the complex tone in Voice n has "fundamental frequency" n, so that its mth partial has frequency $f(m, n) = mn$. Therefore the pitch structure of SC1 can *also* be modeled via a canonical multiplicative array.

In particular, the ratio of the inter-attack duration in voice number n_2 to that in voice number n_1 (ignoring the gradual *accelerando*) is n_2/n_1, while the ratio of their respective fundamental frequencies is also n_2/n_1. In other words, in SC1 the frequency and polyrhythmic ratios between any two given voices are equal. This correspondence is audible at the opening of SC1, where polyrhythms of greater or lesser rhythmic complexity can be discerned involving pitch dyads of respectively greater or lesser harmonic complexity. Inspection of Example 3(b) reveals that this correspondence persists in SC2, although it is difficult to hear amid the more complicated texture. In fact these detailed correspondences become difficult to follow after the first minute of music has elapsed. Be that as it may, coordination between the steady increases in overall harmonic and rhythmic complexity throughout the opening is also a consequence of the structural pitch-rhythm analogy, and this constitutes an *essential* feature of the overall compositional design.

Of course, a stratified polyrhythm does not necessarily depend on Cowell's analogy for its musical significance. If such a texture evokes musical interest, this may instead derive from its inherent perceptible structures or its role within a particular musical context. These have been the primary concerns of this essay. For his part, Tenney indicated that his initial decision to combine "horizontalized" versions of the harmonic series in *Spectral CANON* was motivated not by an interest in pitch-rhythm analogues per se, but by artistic curiosity coupled with an attraction to the series as an abstract structure.[53]

53 Tenney (2003).

4. APPENDIX: FORMAL ANALYSIS OF DIVISIVE ARRAYS

4.1. EQUATIONS OF FLANKING CURVES

If we count from time $x = 0$, the mth attack in Voice n of the canonical divisive array occurs at time-value $x(m,n) = m/n$, as discussed in Section 1 above. Equivalently, a voice whose mth attack occurs at time x has integer-valued voice number $n = m/x$. Thus, for any specified value of m, the curve of the *real-valued* function of a real variable $y(x) = m/x$ passes through the mth attack of each voice. The *attack number*, m, serves as an index into a family of such *flanking curves* about a common vertical asymptote at $x = 0$. Curves farther from the asymptote correspond to greater values of m, as can be corroborated by inspection of Example 4. However, as Example 7 shows, apparently similar families of curves also appear about vertical asymptotes located at other rational time-values. In order to index such curves, we will seek to replace m with an integer-valued index whose absolute value increases away from such an arbitrary asymptote. We will proceed under the assumption that such curves are all vertically scaled and horizontally shifted reciprocal functions. We will subsequently determine which attacks fall on each such curve, in order to verify that they accord with our observations regarding the examples in Section 1.2.

Consider a vertical asymptote located at some arbitrary rational time-value $x_0 = m_0/n_0$, where the fraction m_0/n_0 is in lowest terms.[54] Also consider an attack in Voice n occurring at time $x(m, n) = m/n \neq x_0$. It can be confirmed by direct substitution that the unique real-valued scaled reciprocal function with asymptote x_0 passing through this attack is given by the equation

$$y(x) = \frac{1}{n_0}\frac{j}{x - x_0}, \tag{3}$$

where we have eliminated m by introducing the new non-zero integer-valued index

$$j = n_0 m - m_0 n. \tag{4}$$

The *flanking curve number*, j, serves as an index for a family of such *flanking curves* about the common asymptote $x = x_0$, with curves farther from the asymptote corresponding to greater absolute values of j and curves to the left of the asymptote corresponding to negative values of j. Equation (3) satisfies the differential equation

$$y' = -\frac{n_0}{j}y^2,$$

which shows that, for fixed j and y values, the steepness of flanking curves is proportional to the reduced asymptote denominator n_0.

54 Here $m_0 = 0$ is permitted in order to specify the asymptote at $x = 0$ as a special case.

The only condition imposed on $x(m,n)$ was that it not lie on the asymptote, so any given attack must be located either on that arbitrarily chosen asymptote or on one of its flanking curves (associated with some choice of non-zero integer index j).

4.2. attacks on a flanking curve

The set of points falling on a particular flanking curve can be found by solving the linear Diophantine Equation (4).[55] Since m_0/n_0 is in lowest terms by assumption, it can be shown that for each value of j an infinite family of solutions exists (ignoring, of course, any real-world bounds on the total number of voices), and that individual solutions are of the form

$$(m,n) = (m' + m_0 t, n' + n_0 t) \tag{5}$$

where t is an integer and (m', n') is any single known solution. A given curve may or may not contain an attack corresponding to some particular choice of m, and the same is true for choices of n. Indeed, Equation (5) shows that, for a given flanking curve number j, only one out of every m_0 values of m will be associated with a point on the curve, as will only one out of every n_0 values of n.

If desired, a solution set for the Diophantine equation can also be obtained using the following approach. We say that *a is congruent to b (modulo c)* and write $a \equiv b (\mathrm{mod}\ c)$ if c divides $a - b$. Now simple rearrangements of Equation (4) include $n_0 m - j = m_0 n$ and $m_0 n - (-j) = n_0 m$, so that convenient alternative forms of Equation (4) are the congruences

$$n_0 m \equiv j (\mathrm{mod}\ m_0) \quad \text{and} \quad m_0 n \equiv -j (\mathrm{mod}\ n_0) \tag{6}$$

respectively. The first of these permits straightforward computation of various m values for which there are points on a curve of specified index j, while the second allows a similar computation for n values. Knowing either the m or the n value of some attack on curve j, the other value can be computed using Equation (4), thus furnishing the solution set to the Diophantine equation via Equation (5). The right-hand member of Equations (6) also shows that the pattern of solutions for attacking voice numbers n is periodic in j with period n_0; i.e., the set of voices attacking on each flanking curve, traversing the curves consecutively leftwards or rightwards, displays a cyclical pattern with that period.[56]

4.3. intersections of flanking curves

Consider two flanking curves with indices j_1 and j_2 about unequal asymptotes at m_1/n_1 and m_2/n_2, respectively, and intersecting at a time $x(m,n) = m/n$. Substituting into Equation (3) and solving for (m,n) yields the solution

$$(m,n) = \left(\frac{j_1 m_2 - j_2 m_1}{m_2 n_1 - m_1 n_2}, \frac{j_1 n_2 - j_2 n_1}{m_2 n_1 - m_1 n_2} \right).$$

However, this intersection point only coincides with an attack in the array if its coordinates are positive integers. A special case covering the most musically significant circumstances is the one in which m_1/n_1 and m_2/n_2 are consecutive fractions in a Farey sequence (of whatever order). A general property of such consecutive Farey fractions is that $m_2 n_1 - m_1 n_2 = 1$,[57] so in this case the intersection will occur at

$$(m,n) = (j_1 m_2 - j_2 m_1, j_1 n_2 - j_2 n_1),$$

the coordinates of which are integers so that, as long as they are positive, this intersection corresponds to an attack in the array. The coordinates will *definitely* be positive if $j_1 > 0$ and $j_2 < 0$, implying that every descending curve asymptotic at a Farey fraction intersects at an attack with every ascending curve asymptotic at the succeeding Farey fraction, and vice versa.

4.4. boundaries of metrical regions

Example 16 shows a consecutive-voice subset of a divisive array. Example 16(a) labels selected time-intervals Δx near the instant at which the triple meter associated with asymptote 1/3 gives way to the duple meter associated with the asymptote 1/2. Here Δx_l and Δx_r refer to the time-intervals between adjacent attacks on flanking curves about these respective asymptotes. As time increases away from $x = 1/3$, Δx_l increases while Δx_r decreases. Near $x = 1/3$, attacks sharing a flanking curve about that asymptote are close together in time and likely to form temporal gestalt groups, but when Δx_l ultimately exceeds Δx_r then attacks sharing a flanking curve about $x = 1/2$ will become more likely to group perceptually, marking the transition from one metrical region to the next.

In order to compute the time of the transition, we solve Equation (3) for x yielding

$$x(y) = \frac{j}{n_0 y} + x_0.$$

Now consider a single flanking curve with index j, and two adjacent attacks on it at locations (x_1, y_1) and (x_2, y_2). Then, using the fact that $|y_2 - y_1| = n_0$, the difference in time between these two attacks is

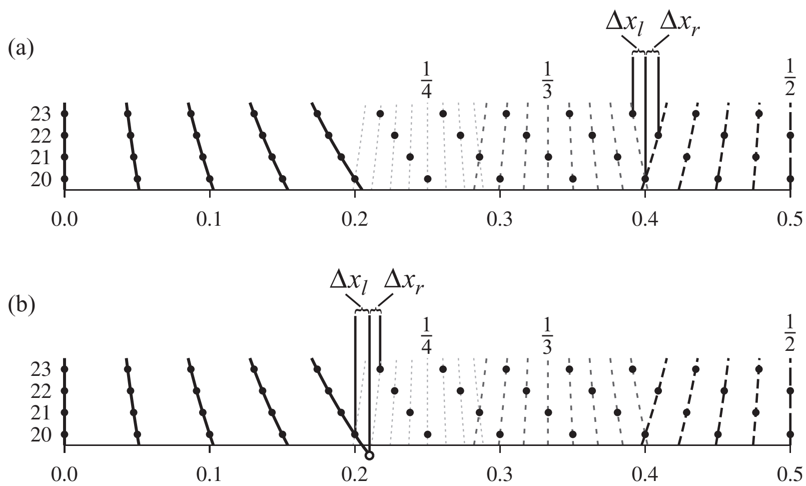
$$|\Delta x| = |x_2 - x_1| = \frac{|j|}{n_0} \left| \frac{1}{y_2} - \frac{1}{y_1} \right| = \frac{|j|}{n_0} \frac{|y_2 - y_1|}{y_1 y_2} = \frac{|j|}{y_1 y_2} \approx \frac{|j|}{\bar{y}^2} \tag{7}$$

where $\bar{y} = (y_1 + y_2)/2$ and where the final approximation is good whenever the number of voices $N' \ll \bar{y}$. Referring to

55 An equation $ax + by = c$ where a, b and c are integers is called a *linear Diophantine equation* if it is to be solved for integer values of x and y. Shockley (1967) discusses the solution of such equations.

56 Since attacks falling *on* the asymptote obey $n \equiv 0$ (mod n_0), the right-hand member of Equations (6) still holds if $j = 0$ is taken to represent the asymptote; i.e., attacks on the asymptote also participate in the cyclical pattern of attacking voices.

57 See Hardy and Wright ([1938] 1979, 23).



EXAMPLE 16 a) AND b). *Transitions between metrical regions in a consecutive-voice subset of a divisive array. The horizontal axis represents time* x, *while the vertical axis represents voice number* n.

Example 16(a) and indexing curves j_l to the right of $x = 1/3$ and j_r to the left of $x = 1/2$, we conclude from Equation (7) that $|\Delta x_r| < |\Delta x_l|$ when $|j_r| < |j_l|$. Thus the metrical transition occurs roughly an equal number of flanking curves away from each of the two asymptotes.

In order to compute the time value at which this transition occurs, we determine the point of intersection between the jth flanking curve to the right of the old asymptote with the jth flanking curve to the left of the new asymptote. Expressing the old and new asymptotes as m_1/n_1 and m_2/n_2, respectively, and using Equation (3), we write

$$\frac{1}{n_1}\frac{j}{x-(m_1/n_1)} = \frac{1}{n_2}\frac{(-j)}{x-(m_2/n_2)}.$$

Solving for x yields

$$x = \frac{m_1+m_2}{n_1+n_2},$$

which is independent of j and equal to the Farey mediant of the two asymptotes. Thus the extent of a metrical region roughly corresponds to the Farey arc about its associated asymptote.

When the denominator of one of the two asymptotes is equal to the number of sounding voices N', a slightly different analysis is required, although a similar conclusion is reached. In such cases, the attack group on each flanking curve about the asymptote in question will comprise just a single sounding attack, so that grouping of attacks on those flanking curves plays no perceptual role. An instance is furnished by Example 16(b) between the asymptotes $x = 1$ and $x = 1/4$, where $n_2 = N' = 4$ and only one attack appears on each flanking curve about $x = 1/4$. In this instance, the onset of perceived quadruple meter occurs roughly when the intragroup and intergroup time-intervals become comparable, so that the four-voice attack cycle becomes sufficiently rhythmically even. This threshold is less well defined perceptually than when attack-group formation is involved, but we may reasonably assume that the transition cannot begin until the intergroup interval is less than twice the intragroup interval. Now for $N' \ll \bar{y}$ the intragroup time-interval between the *arpeggiandi* attacks is $\Delta x_l + \Delta x_r$. (A notional attack in Voice 19 has been added to Example 16[b] in order to illustrate this for the case in which Δx_l and Δx_r are the intergroup time-intervals between attacks on flanking curves about $x = 1$ and $x = 1/4$ respectively.) Thus the minimal requirement for metrical transition is that $\Delta x_l + \Delta x_r < 2\Delta x_l$. Using Equation (7), this condition reduces to $|j_r| < |j_l|$ as before.

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